THE SET OF NONPRINCIPAL ORBITS OF AN ACTION ON E^n

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The purpose of this note is to prove the following fact which was conjectured by Frank Raymond on p. 352 of [**4**].

THEOREM. *Let G be a compact Lie group (not necessarily connected) acting on euclidean space E^n. Suppose that there is at least one stationary point. Then the set of points on nonprincipal orbits is connected.*

(As Raymond notes, this is not hard to prove for abelian groups G. Also, it is known to be false without the hypothesis of the existence of a stationary point, even for $G = Z_6$; see [**3**].)

Actually, this theorem is a fairly easy consequence of the following result, which was essentially proved by us in [**1**, Chapter XV, Lemma 2.7] and [**2**, Lemma 2.2]. It is a generalization of the well-known fact that the orbits of the action of the Weyl group on the Lie algebra of a maximal torus of a compact Lie group G correspond in a one-one manner to the orbits of the adjoint representation of G with the regular points also corresponding.

THEOREM. *Let M^n be either E^n or S^n and let G be a compact Lie group acting on M^n. Let (H) be the principal orbit type and let T be a maximal torus of H. Then the natural map*

$$f: \frac{F(T, M^n)}{N(T)} \to \frac{M^n}{G}$$

is a homeomorphism and takes the set of principal orbits of the $N(T)/T$-action on $F(T, M^n)$ onto those of the action of G on M^n.

Here $N(T)$ is the normalizer of T in G and $F(T, M^n)$ denotes the set of stationary points for T on M^n. Note that the action of $N(T)/T$ on $F(T, M^n)$ may not be effective, but this will not concern us.

Actually the references cited prove this result in a local sense about a fixed point and with M^n any generalized manifold over the integers. However, applying this local result to a neighborhood of the vertex of the open cone over S^n gives the global result for S^n, and passage to the one point compactification gives the global result for E^n.

To prove Raymond's conjecture, we may take 0 to be a stationary point. Let C be the set of points on nonprincipal orbits and let $x \in C$;

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that is, H is conjugate to a proper subgroup of the isotropy group G_x. Without loss of generality we may take H to be a proper subgroup of G_x. Then $x \in F(T, E^n)$ and the $N(T)/T$-orbit of x on $F(T, E^n)$ is not principal for this action. Let $G' = N(T)/T$ and let (H') be the principal orbit type of G' on $F(T, E^n)$. Then we may take H' to be a proper subgroup of G'_x. Note that H' is finite. Now if G'_x is not finite then it contains a circle subgroup S and then $F(S, F(T, E^n))$ is a connected (in fact, acyclic) set containing 0 and x and contained in C. Similarly, suppose that G'_x is finite. Then there is a prime power p^k dividing $\operatorname{ord}(G'_x)$ but not dividing $\operatorname{ord}(H')$. Let P be a p-Sylow subgroup of G'_x. Then, since P is not conjugate to a subgroup of H', we see that $F(P, F(T, E^n))$ is contained in C and, by the Smith theorems, this is a connected set containing 0 and x.

References

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3. J. M. Kister, *Differentiable periodic actions on E^8 without fixed points*, Amer. J. Math. **85** (1963), 316–319.

4. P. S. Mostert, (editor), *Proc. Conference on Transformation Groups*, Springer-Verlag, New York, 1968.

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