ON THE CONVERGENCE OF A SEQUENCE OF PERRON INTEGRALS[1]

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Introduction. This paper is concerned with the convergence of a sequence of Perron integrals. Oscar Perron [4] considered the case of a sequence of uniformly convergent Perron integrable functions, and Bauer [1] extended Perron's work to functions defined in a space of n-dimensions. McShane [3] relaxed the condition of a uniformly convergent sequence of Perron integrable functions and stated necessary conditions for the limit of a sequence of Perron integrals to be the integral of the limit function. The following theorem is a generalization of the above. Throughout this paper integration is in the Perron sense. The Lebesgue integral is denoted by $(\mathfrak{L})\int$.

THEOREM. H1. *$\{f_n(x)\}$ is a sequence of Perron integrable functions whose domain is $I=\{x \mid a\leqq x\leqq b\}$.*

H2. *$f_n(x)\geqq g(x)$ for each n, a.e. (almost everywhere) on I, where $g(x)$ is Perron integrable on I.*

H3. *$\lim_n f_n(x)=f(x)$ a.e. on I.*

Under hypotheses H1–H3, *$f(x)$ is Perron integrable on I, and $\lim_n \int_a^x f_n(t)dt=\int_a^x f(t)dt$, if and only if the sequence of integrals $\{\int_a^x [f_n(t)-g(t)]dt\}$ is EAC (equi-absolutely continuous) on I.*

Preliminary theorems. The following theorems are used in the proof of the theorem above. The reader is referred to Kamke [2] or McShane [3] for a proof of Theorem 1, Theorem 2, and Theorem 3. Vitali [5] gave a proof for Theorem 4.

THEOREM 1. *If each of $f_1(x)$ and $f_2(x)$ is a Perron integrable function on I, and k_1 and k_2 are numbers, then $[k_1f_1(x)+k_2f_2(x)]$ is Perron integrable on I, and*

$$\int_a^x [k_1f_1(t)+k_2f_2(t)]dt = k_1\int_a^x f_1(t)dt + k_2\int_a^x f_2(t)dt.$$

THEOREM 2. *If $f(x)$ is Lebesgue integrable on I, then $f(x)$ is Perron integrable on I and*

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$$\int_a^x f(t)dt = (\mathfrak{L}) \int_a^x f(t)dt.$$

THEOREM 3. *If $f(x)$ is Perron integrable on I, and $f(x) \geqq 0$ a.e. on I, then $f(x)$ is Lebesgue integrable on I.*

THEOREM 4. *If $\{f_n(x)\}$ is a sequence of Lebesgue integrable functions for x on I, $\lim_n f_n(x) = f(x)$ a.e. on I, and $f_n(x) \geqq 0$ a.e. on I, then $f(x)$ is Lebesgue integrable and $\lim_n (\mathfrak{L}) \int_a^x f_n(t)dt = (\mathfrak{L}) \int_a^x f(t)dt$ if and only if the sequence $\{(\mathfrak{L}) \int_a^x f_n(t)dt\}$ is EAC on I.*

PROOF OF THEOREM. (i) *Proof of the Sufficiency.* Since $f_n(x) \geqq g(x)$ a.e. on I, then by Theorem 1, Theorem 2, and Theorem 3 we have for x on I

$$\int_a^x [f_n(t) - g(t)]dt = (\mathfrak{L}) \int_a^x [f_n(t) - g(t)]dt$$

and so the sequence $\{(\mathfrak{L}) \int_a^x [f_n(t) - g(t)]dt\}$ is EAC on I. Then by Theorem 4,

$$\lim_n (\mathfrak{L}) \int_a^x [f_n(t) - g(t)]dt = (\mathfrak{L}) \int_a^x [f(t) - g(t)]dt$$

and Theorem 2 yields

$$\lim_n \int_a^x [f_n(t) - g(t)]dt = \int_a^x [f(t) - g(t)]dt \qquad \text{for } x \text{ on } I.$$

Now, since $g(x)$ is a Perron integrable function on I, then $f(x)$ is Perron integrable on I, and

$$\lim_n \int_a^x f_n(t)dt = \int_a^x f(t)dt.$$

(ii) *Proof of the Necessity.* Under hypothesis,

$$\lim_n \int_a^x f_n(t)dt = \int_a^x f(t)dt.$$

Then by Theorem 1,

$$\lim_n \int_a^x [f_n(t) - g(t)]dt = \int_a^x [f(t) - g(t)]dt$$

and by Theorem 3,

$$\lim \int_a^x [f_n(t) - g(t)]dt = \lim_n (\mathcal{L}) \int_a^x [f_n(t) - g(t)]dt$$

or,

$$\lim_n (\mathcal{L}) \int_a^x [f_n(t) - g(t)]dt = (\mathcal{L}) \int_a^x [f(t) - g(t)]dt.$$

Hence by Theorem 4, the sequence $\{(\mathcal{L})\int_a^x [f_n(t) - g(t)]dt\}$ is EAC on I, and Theorem 2, yields the required result.

References

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