ON THE CONVERGENCE OF A SEQUENCE OF PERRON INTEGRALS

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Introduction. This paper is concerned with the convergence of a sequence of Perron integrals. Oscar Perron [4] considered the case of a sequence of uniformly convergent Perron integrable functions, and Bauer [1] extended Perron's work to functions defined in a space of n-dimensions. McShane [3] relaxed the condition of a uniformly convergent sequence of Perron integrable functions and stated necessary conditions for the limit of a sequence of Perron integrals to be the integral of the limit function. The following theorem is a generalization of the above. Throughout this paper integration is in the Perron sense. The Lebesgue integral is denoted by $(\mathcal{L})\int$.

**Theorem.** H1. $\{f_n(x)\}$ is a sequence of Perron integrable functions whose domain is $I = \{x|a \leq x \leq b\}$.
H2. $f_n(x) \geq g(x)$ for each $n$, a.e. (almost everywhere) on $I$, where $g(x)$ is Perron integrable on $I$.
H3. $\lim_n f_n(x) = f(x)$ a.e. on $I$.

Under hypotheses H1–H3, $f(x)$ is Perron integrable on $I$, and

$$\lim_n \int_a^b f_n(t)dt = \int_a^b f(t)dt$$

if and only if the sequence of integrals

$$\{\int_a^b [f_n(t) - g(t)]dt\}$$

is EAC (equi-absolutely continuous) on $I$.

**Preliminary theorems.** The following theorems are used in the proof of the theorem above. The reader is referred to Kamke [2] or McShane [3] for a proof of Theorem 1, Theorem 2, and Theorem 3. Vitali [5] gave a proof for Theorem 4.

**Theorem 1.** If each of $f_1(x)$ and $f_2(x)$ is a Perron integrable function on $I$, and $k_1$ and $k_2$ are numbers, then $[k_1f_1(x) + k_2f_2(x)]$ is Perron integrable on $I$, and

$$\int_a^b [k_1f_1(t) + k_2f_2(t)]dt = k_1\int_a^b f_1(t)dt + k_2\int_a^b f_2(t)dt.$$

**Theorem 2.** If $f(x)$ is Lebesgue integrable on $I$, then $f(x)$ is Perron integrable on $I$ and

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\[
\int_a^z f(t)\,dt = (\mathcal{L}) \int_a^z f(t)\,dt.
\]

**Theorem 3.** If \( f(x) \) is Perron integrable on \( I \), and \( f(x) \geq 0 \) a.e. on \( I \), then \( f(x) \) is Lebesgue integrable on \( I \).

**Theorem 4.** If \( \{f_n(x)\} \) is a sequence of Lebesgue integrable functions for \( x \) on \( I \), \( \lim_n f_n(x) = f(x) \) a.e. on \( I \), and \( f_n(x) \geq 0 \) a.e. on \( I \), then \( f(x) \) is Lebesgue integrable and \( \lim_n (\mathcal{L}) \int_a^z f(t)\,dt = (\mathcal{L}) \int_a^z f(t)\,dt \) if and only if the sequence \( \{(\mathcal{L}) \int_a^z f_n(t)\,dt\} \) is EAC on \( I \).

**Proof of Theorem.** (i) *Proof of the Sufficiency.* Since \( f_n(x) \geq g(x) \) a.e. on \( I \), then by Theorem 1, Theorem 2, and Theorem 3 we have for \( x \) on \( I \)

\[
\int_a^z [f_n(t) - g(t)]\,dt = (\mathcal{L}) \int_a^z [f_n(t) - g(t)]\,dt
\]

and so the sequence \( \{(\mathcal{L}) \int_a^z [f_n(t) - g(t)]\,dt\} \) is EAC on \( I \). Then by Theorem 4,

\[
\lim_n (\mathcal{L}) \int_a^z [f_n(t) - g(t)]\,dt = (\mathcal{L}) \int_a^z [f(t) - g(t)]\,dt
\]

and Theorem 2 yields

\[
\lim_n \int_a^z [f_n(t) - g(t)]\,dt = \int_a^z [f(t) - g(t)]\,dt \quad \text{for } x \text{ on } I.
\]

Now, since \( g(x) \) is a Perron integrable function on \( I \), then \( f(x) \) is Perron integrable on \( I \), and

\[
\lim_n \int_a^z f_n(t)\,dt = \int_a^z f(t)\,dt.
\]

(ii) *Proof of the Necessity.* Under hypothesis,

\[
\lim_n \int_a^z f_n(t)\,dt = \int_a^z f(t)\,dt.
\]

Then by Theorem 1,

\[
\lim_n \int_a^z [f_n(t) - g(t)]\,dt = \int_a^z [f(t) - g(t)]\,dt
\]

and by Theorem 3,
\[
\lim \int_a^x [f_n(t) - g(t)] dt = \lim_\mathcal{L} \int_a^x [f_n(t) - g(t)] dt
\]
or,
\[
\lim_\mathcal{L} \int_a^x [f_n(t) - g(t)] dt = (\mathcal{L}) \int_a^x [f(t) - g(t)] dt.
\]

Hence by Theorem 4, the sequence \( \{ (\mathcal{L}) \int_a^x [f_n(t) - g(t)] dt \} \) is EAC on \( I \), and Theorem 2, yields the required result.

References

4. O. Perron, Über den Integralbegriff, Sitzungsberichte der Heidelberger Akademie der Wissenschaften, Abhandlung A. Abhandlung 16 (1914).

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