PROPERTIES OF TWO POINT BOUNDARY VALUE FUNCTIONS[1]

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1. Introduction. This paper is concerned with conditions under which certain functions arising from two point boundary value problems of quasi-differential equations are continuous and increasing.

We now define recursively an nth order linear differential operator by:

(1) $\mathcal{K}_0$ is the set of all real-valued continuous functions on $(-\infty, \infty)$ where $D_0(y) = y$ for y an element of $\mathcal{K}_0$, and

(2) for $1 \leqq k \leqq n$, let $\mathcal{K}_k$ be the set of all y in $\mathcal{K}_{k-1}$ which has a continuous derivative and y an element of $\mathcal{K}_k$

$$D_k(y) = \frac{1}{f_{k,k+1}}\left[(D_{k-1}(y))' + \sum_{i=1}^{k-1} p_{k,i}D_{i-1}(y)\right]$$

where $f_{k,k+1}$ is a continuous real-valued positive function on $(-\infty, \infty)$ and $p_{k,i}$ is a continuous function on $(-\infty, \infty)$.

The nth order quasi-differential equation to be studied is

$$(E_n) \qquad L_n[y] = (D_{n-1}(y))' + \sum_{i=1}^{n-1} p_{n,i}D_{i-1}(y) = 0.$$

Quasi-differential equations similar to (E_n) have also been studied by D. Hinton [1] and A. Zettl [6].

Before proceeding further we define the boundary value functions.

DEFINITION 1.1. (a) For $t \in [a, \infty)$, and $i+j \geqq n$ define $z_{ij}(t)$ to be the greatest lower bound of the set of all $b > t$ such that (E_n) has a nontrivial solution with a zero at t of multiplicity at least i and a zero at b of multiplicity at least j.

(b) If $t \in [a, \infty)$ and $i+j \geqq n$ then $r_{ij}(t)$ is the greatest lower bound of the set of all $b > t$ such that there exists a nontrivial solution of (E_n) which has a zero at some $t_0 \geqq t$ of multiplicity at least i and a zero at $b > t_0$ of multiplicity at least j.

DEFINITION 1.2. For $t(-\infty, c]$ and $i+j \geqq n$, then $\bar{z}_{ij}(t)$ is the least upper bound of the set of all $b < t$ such that there exists a non-

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trivial solution y, which has a zero of multiplicity at least i at t and a zero at b of multiplicity at least j.

Finally we define a class of operators which will be useful.

DEFINITION 1.3. If J is a subinterval of $(-\infty, \infty)$ denote by $T_{ij}[J]$ the class of operators L_n for which (E_n) has no nontrivial solution with at least i zero at some $a \in J$ and at least j zeros at some $b > a$, $b \in J$, $(i+j \geqq n)$. If J is $(-\infty, \infty)$ we omit J.

2. **Properties** $z_{ij}(t)$ **and** $r_{ij}(t)$.

DEFINITION 2.1. An extended real-valued function α is increasing if and only if, for any s_1 and s_2 such that $-\infty < s_1 < s_2 < \infty$, either $\alpha(s_1) < \alpha(s_2)$ or $\alpha(s_2) = \infty$.

T. Sherman [5] and D. Hinton [1] showed that $\eta_1(t)$, the first conjugate point of t_1 is an increasing function. J. Levin [2] stated without proof some conditions under which $r_{ij}(t)$ for the classical nth order equation is increasing. We now establish conditions under which $r_{ij}(t)$ and $z_{ij}(t)$ are increasing for (E_n).

LEMMA 2.1. *If either $z_{ij}(t)$ or $r_{ij}(t)$ is an increasing function of t, then $z_{ij}(t) = r_{ij}(t)$, where $i+j=n$.*

The proof of the lemma is a direct consequence of Definition 1.1 and the fact that the greatest lower bound can be replaced by the word minimum.

DEFINITION 2.2. A fundamental set of solution $\{u_\beta(x, t)\}$ of (E_n) is defined by $D_\alpha(u_\beta(t, t)) = \delta_{\alpha+1,\beta+1}$ $(\alpha, \beta = 0, \cdots, n-1)$. We also use the following notation:

$$W(u_i(x, t), \cdots, u_j(x, t)) = \left| D_\alpha[u_\beta(x, t)] \right|, \quad i \leqq \beta \leqq j,\ 0 \leqq \alpha \leqq j-1.$$

When L_{2n} is a selfadjoint operator, the next theorem shows that $z_{nn}(t)$ is an increasing function without placing further restrictions on L_{2n}.

THEOREM 2.1. *If $L_{2n}[y] = 0$ is a selfadjoint equation, then $z_{nn}(t)$ is increasing.*

PROOF. If $Z_{nn}(\alpha) < \infty$, then $z_{nn}(\beta) \leqq z_{nn}(\alpha)$ for $\beta < \alpha$ by Theorem 2.1 of W. T. Reid [4]. Assume now there exists $b > a$ such that $z_{nn}(a) = z_{nn}(b) < \infty$. Then $z_{nn}(\alpha) = z_{nn}(a)$ for $\alpha \in [a, b]$ by the above and $W(u_n(x, z_{nn}(a)), \cdots, u_{2n-1}(x, z_{nn}(a))) = 0$ for $x \in [a, b]$; but by Theorem 3.4 of D. Hinton [1] this cannot happen.

COROLLARY 2.1. *If $L_{2n}[y] = 0$ is a $2n$th order selfadjoint equation then $\bar{z}_{nn}(t)$ is increasing.*

Let $M_+(M_-)$ be the set of all t such that $z_{nn}(t)(\bar{z}_{nn}(t))$ is finite.

COROLLARY 2.2. *For a selfadjoint 2nth order equation* (E_n), $z_{nn}(t)$ *restricted to* M_+ *is continuous.*

The proof follows exactly the same reasoning as in the proof of Corollary 3.5.2 of D. Hinton [1].

Now we consider conditions under which $r_{ij}(t)$ is increasing and continuous for the more general nth order operator L_n.

LEMMA 2.2. *If* $L_n \in T_{k+1,n-k} \cap T_{k-1,n-k+1}$ *for some* $k \in \{1, 2, \cdots, n-1\}$, $\tau < t$ *and* $W(u_{n-k}(\tau, t), \cdots, u_{n-1}(\tau, t)) = 0$, *then*

$$W'(u_{n-k}(\tau, t), \cdots, u_{n-1}(\tau, t)) \neq 0.$$

PROOF. Define constants c_{n-k+i} by

$$c_{n-k+i} = (-1)^i W(u_{n-k}(\tau, t), \cdots, u_{n-k+i-1}(\tau, t), u_{n-k+i+1}(\tau, t), \cdots, u_{n-1}(\tau, t)) \qquad (i = 1, \cdots, k-1).$$

Set

$$y(x) = \sum_{i=n-k}^{n-1} c_i u_i(x, t). \tag{2.1}$$

Since $L_n \in T_{k-1,n-k+1}$, $c_{n-k} = W(u_{n-k+1}(\tau, t), \cdots, u_{n-1}(\tau, t)) \neq 0$ and the function y given by (2.1) is a nontrivial solution of (E_n). We observe now that

$$D_l y(\tau) = \sum_{i=n-k}^{n-1} c_i D_l u_i(\tau, t) = 0, \qquad 0 \leqq l \leqq k-2 \tag{2.2}$$

since $\sum_{i=n-k}^{n-1} c_i D_l u_i(\tau, t)$ is a determinant with two rows the same. Moreover $D_{k-1} y(\tau) = 0$ since $D_{k-1} y(\tau) = W(u_{n-k}(\tau, t), \cdots, u_{n-1}(\tau, t))$. For $1 \leqq k \leqq n-1$ we see that $W'(u_{n-k}(\tau, t), \cdots, u_{n-k}(\tau, t)) \neq 0$, otherwise $D_k(y(\tau)) = 0$ which is a contradiction of the fact that $L \in T_{k+1,n-k}$.

COROLLARY 2.3. *If* L *is selfadjoint* $L \in T_{k-1,n-k} \cap T_{k-1,n-k+1}$ *for some* k, $\tau > t$ *and* $W(u_{n-k}(\tau, t), \cdots, u_{n-1}(\tau, t)) = 0$ *then* $W'(u_{n-k}(\tau, t), \cdots, u_{n-1}(\tau, t)) \neq 0$ *for* $\tau > t$.

The proof follows easily from Lemma 2.2 and Theorem 4.2 of D. B. Hinton [1].

DEFINITION 2.3. Denote by $N_{k,n-k}$ the class of operators for which $r_{k,n-k}(t) < \min(r_{k+1,n-k}(t), r_{k-1,n-k+1}(t))$ if $r_{k,n-k}(t)$ is finite, $-\infty < t < \infty$.

THEOREM 2.2. *If* L_n *is an element of* $N_{k,n-k}$ *for some* $k \in \{1, 2, \cdots, n-1\}$ *then* $r_{k,n-k}(t)$ *is an increasing function of* t.

PROOF. First we observe that $r_{k,n-k}(t)$ is nondecreasing by definition. If there exists γ in $(-\infty, \infty)$ such that $r_{k,n-k}(\gamma)=\infty$ then $r_{k,n-k}(t)$ is increasing on $[\gamma, \infty]$. Also if $c>d$ where $r_{k,n-k}(d)<\infty$ and $r_{k,n-k}(c)=\infty$, then $r_{k,n-k}(d)<r_{k,n-k}(c)$.

In order to complete the proof we need only show $z_{k,n-k}(t) = r_{k,n-k}(t)$, when $r_{k,n-k}(t)<\infty$. From the definition and the fact that greatest lower bound can be replaced by minimum we know there exists a nontrivial solution y of L which has a zero of multiplicity k at $\tau\in[t, r_{k,n-k}(t))$ and a zero of multiplicity $n-k$ at $r_{k,n-k}(t)$. In order to simplify notation we will use β for $r_{k,n-k}(t)$ in the remainder of the proof. This solution y can be written as

$$y(x) = c_{n-k}u_{n-k}(x, \beta) + \cdots + c_{n-1}u_{n-1}(x, \beta),$$

and since y has a zero of multiplicity k at τ,

$$W(u_{n-k}(\tau, \beta), \cdots, u_{n-1}(\tau, \beta))=0.$$

It follows from the continuous dependence of solutions upon initial values and the fact that a determinate is a continuous function of its elements that given $\epsilon>0$ and $i\in\{0, 1, \cdots, n-1\}$, then there exists $\delta_\epsilon>0$ such that $0<\beta-\beta_\epsilon<\delta_\epsilon$ implies that

$$\left| W(u_i(x, \beta), \cdots, u_{n-1}(x, \beta)) - W(u_i(x, \beta_\epsilon), \cdots, u_{n-1}(x, \beta_\epsilon))\right| < \epsilon$$

for all $x\in[t, \beta]$.

Now for $\tau\neq t$, choose $\epsilon<(t-\tau)/2$, $\delta=\min\{\delta_\epsilon, (\tau-t)/4, (\beta-t)/4, \epsilon\}$ and $\beta_\epsilon\in(\beta-\delta, \delta)$. Since $r_{k,n-k}(t)<\min(r_{k+1,n-k}(t), r_{k-1,n-k+1}(t))$ then L is an element of $T_{k+1,n-k}\cap T_{k-1,n-k+1}$ on $[t, \beta]$. By Lemma 2.2, we know that $W'(u_{n-k}(\tau, \beta), \cdots, u_{n-1}(\tau, \beta))\neq 0$. But

$$W(u_{n-k}(\tau, \beta), \cdots, u_{n-1}(\tau, \beta)) = 0$$

and thus it follows that for some $x\in(\tau-\epsilon, \tau+\epsilon)$

$$W(u_{n-k}(x, \beta_\epsilon), \cdots, u_{n-1}(x, \beta_\epsilon)) = 0.$$

Hence there exists a nontrivial solution y, which has a zero of order $n-k$ at $\beta_\epsilon<\beta$ and a zero of order k at a point $\tau>t$, which is a contradiction. Hence $t=\tau$ and the proof is complete.

COROLLARY 2.4. *Let* $L[y]=L_0[y]+py$ *where* $L_0\in T_0$ *and* $p(t) >0$ (<0). *Then* $r_{k,n-k}(t)$ *is increasing.*

The proof follows from Theorem 2.3 and Theorem 3.2 of Z. Nehari [**3**].

By strengthening the hypothesis of Theorem 2.3 we now give a condition under which $r_{k,n-k}(t)$ is both increasing and continuous.

First let $S_k = \{t: r_{k,n-k}(t) \text{ is finite}\}$.

THEOREM 2.3. *If* $r_{k,n-k}(t) < \min\{r_{k+1,n-k}(t), r_{k-1,n-k+1}(t)\}$ *for some* $k \in \{1, 2, \cdots, n-1\}$, *then* S_k *is an interval and* $r_{k,n-k}(t)$ *is continuous on* S_k.

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