

# PROPERTIES OF TWO POINT BOUNDARY VALUE FUNCTIONS<sup>1</sup>

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**1. Introduction.** This paper is concerned with conditions under which certain functions arising from two point boundary value problems of quasi-differential equations are continuous and increasing.

We now define recursively an  $n$ th order linear differential operator by:

(1)  $\mathcal{C}_0$  is the set of all real-valued continuous functions on  $(-\infty, \infty)$  where  $D_0(y) = y$  for  $y$  an element of  $\mathcal{C}_0$ , and

(2) for  $1 \leq k \leq n$ , let  $\mathcal{C}_k$  be the set of all  $y$  in  $\mathcal{C}_{k-1}$  which has a continuous derivative and  $y$  an element of  $\mathcal{C}_k$

$$D_k(y) = \frac{1}{f_{k,k+1}} \left[ (D_{k-1}(y))' + \sum_{i=1}^{k-1} p_{k,i} D_{i-1}(y) \right]$$

where  $f_{k,k+1}$  is a continuous real-valued positive function on  $(-\infty, \infty)$  and  $p_{k,i}$  is a continuous function on  $(-\infty, \infty)$ .

The  $n$ th order quasi-differential equation to be studied is

$$(E_n) \quad L_n[y] = (D_{n-1}(y))' + \sum_{i=1}^{n-1} p_{n,i} D_{i-1}(y) = 0.$$

Quasi-differential equations similar to  $(E_n)$  have also been studied by D. Hinton [1] and A. Zettl [6].

Before proceeding further we define the boundary value functions.

**DEFINITION 1.1.** (a) For  $t \in [a, \infty)$ , and  $i+j \geq n$  define  $z_{ij}(t)$  to be the greatest lower bound of the set of all  $b > t$  such that  $(E_n)$  has a nontrivial solution with a zero at  $t$  of multiplicity at least  $i$  and a zero at  $b$  of multiplicity at least  $j$ .

(b) If  $t \in [a, \infty)$  and  $i+j \geq n$  then  $r_{ij}(t)$  is the greatest lower bound of the set of all  $b > t$  such that there exists a nontrivial solution of  $(E_n)$  which has a zero at some  $t_0 \geq t$  of multiplicity at least  $i$  and a zero at  $b > t_0$  of multiplicity at least  $j$ .

**DEFINITION 1.2.** For  $t \in (-\infty, c]$  and  $i+j \geq n$ , then  $\bar{z}_{ij}(t)$  is the least upper bound of the set of all  $b < t$  such that there exists a non-

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trivial solution  $y$ , which has a zero of multiplicity at least  $i$  at  $t$  and a zero at  $b$  of multiplicity at least  $j$ .

Finally we define a class of operators which will be useful.

**DEFINITION 1.3.** If  $J$  is a subinterval of  $(-\infty, \infty)$  denote by  $T_{ij}[J]$  the class of operators  $L_n$  for which  $(E_n)$  has no nontrivial solution with at least  $i$  zero at some  $a \in J$  and at least  $j$  zeros at some  $b > a$ ,  $b \in J$ ,  $(i+j \geq n)$ . If  $J$  is  $(-\infty, \infty)$  we omit  $J$ .

## 2. Properties $z_{ij}(t)$ and $r_{ij}(t)$ .

**DEFINITION 2.1.** An extended real-valued function  $\alpha$  is increasing if and only if, for any  $s_1$  and  $s_2$  such that  $-\infty < s_1 < s_2 < \infty$ , either  $\alpha(s_1) < \alpha(s_2)$  or  $\alpha(s_2) = \infty$ .

T. Sherman [5] and D. Hinton [1] showed that  $\eta_1(t)$ , the first conjugate point of  $t_1$  is an increasing function. J. Levin [2] stated without proof some conditions under which  $r_{ij}(t)$  for the classical  $n$ th order equation is increasing. We now establish conditions under which  $r_{ij}(t)$  and  $z_{ij}(t)$  are increasing for  $(E_n)$ .

**LEMMA 2.1.** *If either  $z_{ij}(t)$  or  $r_{ij}(t)$  is an increasing function of  $t$ , then  $z_{ij}(t) = r_{ij}(t)$ , where  $i+j = n$ .*

The proof of the lemma is a direct consequence of Definition 1.1 and the fact that the greatest lower bound can be replaced by the word minimum.

**DEFINITION 2.2.** A fundamental set of solution  $\{u_\beta(x, t)\}$  of  $(E_n)$  is defined by  $D_\alpha(u_\beta(t, t)) = \delta_{\alpha+1, \beta+1}$  ( $\alpha, \beta = 0, \dots, n-1$ ). We also use the following notation:

$$W(u_i(x, t), \dots, u_j(x, t)) = |D_\alpha[u_\beta(x, t)]|, \quad i \leq \beta \leq j, \quad 0 \leq \alpha \leq j-1.$$

When  $L_{2n}$  is a selfadjoint operator, the next theorem shows that  $z_{nn}(t)$  is an increasing function without placing further restrictions on  $L_{2n}$ .

**THEOREM 2.1.** *If  $L_{2n}[y] = 0$  is a selfadjoint equation, then  $z_{nn}(t)$  is increasing.*

**PROOF.** If  $Z_{nn}(\alpha) < \infty$ , then  $z_{nn}(\beta) \leq z_{nn}(\alpha)$  for  $\beta < \alpha$  by Theorem 2.1 of W. T. Reid [4]. Assume now there exists  $b > a$  such that  $z_{nn}(a) = z_{nn}(b) < \infty$ . Then  $z_{nn}(\alpha) = z_{nn}(a)$  for  $\alpha \in [a, b]$  by the above and  $W(u_n(x, z_{nn}(a)), \dots, u_{2n-1}(x, z_{nn}(a))) = 0$  for  $x \in [a, b]$ ; but by Theorem 3.4 of D. Hinton [1] this cannot happen.

**COROLLARY 2.1.** *If  $L_{2n}[y] = 0$  is a  $2n$ th order selfadjoint equation then  $\bar{z}_{nn}(t)$  is increasing.*

Let  $M_+(M_-)$  be the set of all  $t$  such that  $z_{nn}(t)(\bar{z}_{nn}(t))$  is finite.

**COROLLARY 2.2.** *For a selfadjoint 2nth order equation  $(E_n)$ ,  $z_{nn}(t)$  restricted to  $M_+$  is continuous.*

The proof follows exactly the same reasoning as in the proof of Corollary 3.5.2 of D. Hinton [1].

Now we consider conditions under which  $r_{ij}(t)$  is increasing and continuous for the more general  $n$ th order operator  $L_n$ .

**LEMMA 2.2.** *If  $L_n \in T_{k+1, n-k} \cap T_{k-1, n-k+1}$  for some  $k \in \{1, 2, \dots, n-1\}$ ,  $\tau < t$  and  $W(u_{n-k}(\tau, t), \dots, u_{n-1}(\tau, t)) = 0$ , then*

$$W'(u_{n-k}(\tau, t), \dots, u_{n-1}(\tau, t)) \neq 0.$$

**PROOF.** Define constants  $c_{n-k+i}$  by

$$c_{n-k+i} = (-1)^i W(u_{n-k}(\tau, t), \dots, u_{n-k+i-1}(\tau, t), u_{n-k+i+1}(\tau, t), \dots, u_{n-1}(\tau, t)) \\ (i = 1, \dots, k-1).$$

Set

$$(2.1) \quad y(x) = \sum_{i=n-k}^{n-1} c_i u_i(x, t).$$

Since  $L_n \in T_{k-1, n-k+1}$ ,  $c_{n-k} = W(u_{n-k+1}(\tau, t), \dots, u_{n-1}(\tau, t)) \neq 0$  and the function  $y$  given by (2.1) is a nontrivial solution of  $(E_n)$ . We observe now that

$$(2.2) \quad D_l y(\tau) = \sum_{i=n-k}^{n-1} c_i D_l u_i(\tau, t) = 0, \quad 0 \leq l \leq k-2$$

since  $\sum_{i=n-k}^{n-1} c_i D_l u_i(\tau, t)$  is a determinant with two rows the same. Moreover  $D_{k-1} y(\tau) = 0$  since  $D_{k-1} y(\tau) = W(u_{n-k}(\tau, t), \dots, u_{n-1}(\tau, t))$ . For  $1 \leq k \leq n-1$  we see that  $W'(u_{n-k}(\tau, t), \dots, u_{n-k}(\tau, t)) \neq 0$ , otherwise  $D_k(y(\tau)) = 0$  which is a contradiction of the fact that  $L \in T_{k+1, n-k}$ .

**COROLLARY 2.3.** *If  $L$  is selfadjoint  $L \in T_{k-1, n-k} \cap T_{k-1, n-k+1}$  for some  $k$ ,  $\tau > t$  and  $W(u_{n-k}(\tau, t), \dots, u_{n-1}(\tau, t)) = 0$  then  $W'(u_{n-k}(\tau, t), \dots, u_{n-1}(\tau, t)) \neq 0$  for  $\tau > t$ .*

The proof follows easily from Lemma 2.2 and Theorem 4.2 of D. B. Hinton [1].

**DEFINITION 2.3.** Denote by  $N_{k, n-k}$  the class of operators for which  $r_{k, n-k}(t) < \min(r_{k+1, n-k}(t), r_{k-1, n-k+1}(t))$  if  $r_{k, n-k}(t)$  is finite,  $-\infty < t < \infty$ .

**THEOREM 2.2.** *If  $L_n$  is an element of  $N_{k, n-k}$  for some  $k \in \{1, 2, \dots, n-1\}$  then  $r_{k, n-k}(t)$  is an increasing function of  $t$ .*

PROOF. First we observe that  $r_{k,n-k}(t)$  is nondecreasing by definition. If there exists  $\gamma$  in  $(-\infty, \infty)$  such that  $r_{k,n-k}(\gamma) = \infty$  then  $r_{k,n-k}(t)$  is increasing on  $[\gamma, \infty]$ . Also if  $c > d$  where  $r_{k,n-k}(d) < \infty$  and  $r_{k,n-k}(c) = \infty$ , then  $r_{k,n-k}(d) < r_{k,n-k}(c)$ .

In order to complete the proof we need only show  $z_{k,n-k}(t) = r_{k,n-k}(t)$ , when  $r_{k,n-k}(t) < \infty$ . From the definition and the fact that greatest lower bound can be replaced by minimum we know there exists a nontrivial solution  $y$  of  $L$  which has a zero of multiplicity  $k$  at  $\tau \in [t, r_{k,n-k}(t))$  and a zero of multiplicity  $n - k$  at  $r_{k,n-k}(t)$ . In order to simplify notation we will use  $\beta$  for  $r_{k,n-k}(t)$  in the remainder of the proof. This solution  $y$  can be written as

$$y(x) = c_{n-k}u_{n-k}(x, \beta) + \dots + c_{n-1}u_{n-1}(x, \beta),$$

and since  $y$  has a zero of multiplicity  $k$  at  $\tau$ ,

$$W(u_{n-k}(\tau, \beta), \dots, u_{n-1}(\tau, \beta)) = 0.$$

It follows from the continuous dependence of solutions upon initial values and the fact that a determinate is a continuous function of its elements that given  $\epsilon > 0$  and  $i \in \{0, 1, \dots, n-1\}$ , then there exists  $\delta_\epsilon > 0$  such that  $0 < \beta - \beta_\epsilon < \delta_\epsilon$  implies that

$$|W(u_i(x, \beta), \dots, u_{n-1}(x, \beta)) - W(u_i(x, \beta_\epsilon), \dots, u_{n-1}(x, \beta_\epsilon))| < \epsilon$$

for all  $x \in [t, \beta]$ .

Now for  $\tau \neq t$ , choose  $\epsilon < (t - \tau)/2$ ,  $\delta = \min\{\delta_\epsilon, (\tau - t)/4, (\beta - t)/4, \epsilon\}$  and  $\beta_\epsilon \in (\beta - \delta, \delta)$ . Since  $r_{k,n-k}(t) < \min(r_{k+1,n-k}(t), r_{k-1,n-k+1}(t))$  then  $L$  is an element of  $T_{k+1,n-k} \cap T_{k-1,n-k+1}$  on  $[t, \beta]$ . By Lemma 2.2, we know that  $W'(u_{n-k}(\tau, \beta), \dots, u_{n-1}(\tau, \beta)) \neq 0$ . But

$$W(u_{n-k}(\tau, \beta), \dots, u_{n-1}(\tau, \beta)) = 0$$

and thus it follows that for some  $x \in (\tau - \epsilon, \tau + \epsilon)$

$$W(u_{n-k}(x, \beta_\epsilon), \dots, u_{n-1}(x, \beta_\epsilon)) = 0.$$

Hence there exists a nontrivial solution  $y$ , which has a zero of order  $n - k$  at  $\beta_\epsilon < \beta$  and a zero of order  $k$  at a point  $\tau > t$ , which is a contradiction. Hence  $t = \tau$  and the proof is complete.

COROLLARY 2.4. *Let  $L[y] = L_0[y] + py$  where  $L_0 \in T_0$  and  $p(t) > 0$  ( $< 0$ ). Then  $r_{k,n-k}(t)$  is increasing.*

The proof follows from Theorem 2.3 and Theorem 3.2 of Z. Nehari [3].

By strengthening the hypothesis of Theorem 2.3 we now give a condition under which  $r_{k,n-k}(t)$  is both increasing and continuous.

First let  $S_k = \{t: r_{k,n-k}(t) \text{ is finite}\}$ .

**THEOREM 2.3.** *If  $r_{k,n-k}(t) < \min\{r_{k+1,n-k}(t), r_{k-1,n-k+1}(t)\}$  for some  $k \in \{1, 2, \dots, n-1\}$ , then  $S_k$  is an interval and  $r_{k,n-k}(t)$  is continuous on  $S_k$ .*

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