Let $G$ be a locally compact Hausdorff group. Let $C_0(G)$ denote the set of all complex-valued continuous functions on $G$ such that for each $\varepsilon > 0$, there exists a compact subset $K$ of $G$ such that $|f(x)| < \varepsilon$ for all $x \in K$. Let $C_00(G)$ be the subset consisting of functions in $C_0(G)$ with compact support. Suppose $V(G)$ be the set of all real or complex regular Borel measures on $G$ which are finite on compact sets. The subset $M(G)$ which have finite total variation is the dual of $C_0(G)$. For $\mu \in V(G)$ and $s \in G$ let $\mu_s$ be the measure in $V(G)$ defined by $\mu_s(E) = \mu(Es^{-1})$ for each Borel set $E$. A measure $\mu$ is said to have separable orbit if there is a countable subset $C \subseteq G$ with the property that $s \in G$ and $\varepsilon > 0$ there exists a $c \in C$ such that $|\mu_s - \mu_c|(G) = \|\mu_s - \mu_c\| < \varepsilon$.

In Theorem 2 of [2, p. 570], it was proved that a measure with separable orbit is absolutely continuous if $G$ is second countable. We shall give a proof of this result without the second countability of $G$. When $G$ is not $\sigma$-compact, $\mu \in V(G)$ has separable orbit if and only if $G$ contains an open-closed subgroup $H$ whose index is at most countable and $\mu$ on each coset of $H$ is some multiple of the Haar measure on that set.

We begin with the following lemma which has been proved in [2, p. 570] under the condition that $G$ is second countable. We give a different argument without this restriction.

**Lemma.** For every $\mu$ in $V(G)$ which has separable orbit, $\lim_{s \to c}\|\mu_s - \mu_c\| = 0$. Thus if $\mu$ is in $M(G)$, the function $s \mapsto \mu_s$ is continuous.

**Proof.** For each integer $m > 0$,

$$G = \bigcup_{c \in C} \{x \in G: \|\mu_x - \mu_c\| \leq 1/m\}.$$

Let $x_\alpha$ be in $S_\varepsilon = \{x: \|\mu_x - \mu_c\| \leq 1/m\}$ and converge to $x$. Since the function $f(x_\alpha)$ converges to $f(x)$ uniformly with every $f$ in $C_00(G)$,

$$\lim \int f(s) d\mu_{x_\alpha}(s) = \lim \int f(sx_\alpha) d\mu(s) = \int f(s) d\mu_x(s).$$

Thus $\int f(s) d(\mu_x - \mu_c) = \lim \int f(s) d(\mu_{x_\alpha} - \mu_c) \leq \lim \sup \|\mu_{x_\alpha} - \mu_c\| \|f\|_\infty \leq \|f\|_\infty /m$ for all $f \in C_00(G)$. So $\|\mu_x - \mu_c\| \leq 1/m$, and $S_\varepsilon$ is closed. As
G is locally compact Hausdorff, it is of second category and there is a \( c_m \) so that \( S_m = \{ x : \| \mu_x - \mu_{c_m} \| \leq 1/m \} \) has interior. Then there exists a neighborhood \( U_m \) of the identity of \( G \), such that \( x_m U_m \subseteq S_m \). Now for each \( s \in U_m \), we have
\[
\| \mu_s - \mu_t \| = \| \mu_s x_m - \mu_{c_m} \| \leq \| \mu_s x_m - \mu_s c_m \| + \| \mu_{c_m} - \mu_{c_m} \| \leq 2/m.
\]
Thus \( \lim_{s \to e} \| \mu_s - \mu \| = 0 \).

**Theorem 1.** Every measure \( \mu \) in \( V(G) \) which has a separable orbit is absolutely continuous.

**Proof.** For every compact set \( K \) in \( G \) we have
\[
| \mu_s - \mu_t | (K) \leq \| \mu_s - \mu_t \|. 
\]
Hence \( \mu_t(K) \) is continuous as a function of \( t \) and \( \mu \) is absolutely continuous [3, p. 230].

If \( G \) is not \( \sigma \)-compact, then it has an open-closed subgroup \( K \) such that \( G/K \) is uncountable [1, p. 34]. Every \( \mu \in M(G) \) is supported on a countable union of cosets of \( K \). So the orbit of \( \mu \) cannot be separable unless \( \mu = 0 \). We wish to thank the referee for the following:

**Theorem 2.** Suppose \( G \) is not \( \sigma \)-compact. Then \( \mu \in V(G) \) has separable orbit if and only if there is an open-closed subgroup \( H \) of \( G \) whose index in \( G \) is at most countable, and such that the restriction of \( \mu \) to each coset of \( H \) is a multiple of the Haar measure on that set. In particular, if \( \mu \) has separable orbit, then its orbit is at most countable.

**Proof.** Let \( H \) be the set \( \{ s \in G : \| \mu_s - \mu \| < \infty \} \). For any \( s, t \in H \),
\[
\| \mu_s t^{-1} - \mu_t \| \leq \| \mu_s - \mu_t \| + \| \mu_t - \mu \| < \infty.
\]
Hence \( H \) is a subgroup. By the Lemma, \( H \) is open and thus closed. Since the orbit of \( \mu \) is separable, \( H \) has at most countable index in \( G \). Moreover, \( H \) is not \( \sigma \)-compact. For all \( t \in H \), \( \mu_t - \mu \in M(G) \) has a separable orbit. So \( \mu_t - \mu = 0 \) and \( \mu \) is invariant under translation by elements in \( H \). Thus \( \mu \) on coset of \( H \) must be a multiple of Haar measure on that set. The converse is clear.

**Theorem 3.** If \( G \) is \( \sigma \)-compact, and \( \mu \in V(G) \) is such that \( \lim_{s \to e} \| \mu_s - \mu \| = 0 \), then \( \mu \) has separable orbit. In particular, \( \mu \in M(G) \) is absolutely continuous if and only if it has separable orbit.

**Proof.** Again let \( H \) be the set \( \{ s \in G : \| \mu_s - \mu \| < \infty \} \). As in the proof of Theorem 2, \( H \) is an open-closed subgroup. Since \( G \) is \( \sigma \)-compact, it has the Lindelöf property. The index of \( H \) in \( G \) is at most countable. Now the map \( s \to \mu_s - \mu \) on \( H \) is continuous in the total variation norm. So the set \( \{ \mu_s - \mu : s \in H \} \) is \( \sigma \)-compact and thus separable. It follows
that the orbit of \( \mu \) under translation by elements of \( H \) is separable. Since the index of \( H \) in \( G \) is at most countable, the orbit of \( \mu \) is separable.

**Bibliography**


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