ON MEASURES WITH SEPARABLE ORBIT

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Let G be a locally compact Hausdorff group. Let $C_0(G)$ denote the set of all complex-valued continuous functions on G such that for each $\epsilon>0$, there exists a compact subset K of G such that $|f(x)|<\epsilon$ for all $x\notin K$. Let $C_{00}(G)$ be the subset consisting of functions in $C_0(G)$ with compact support. Suppose $V(G)$ be the set of all real or complex regular Borel measures on G which are finite on compact sets. The subset $M(G)$ which have finite total variation is the dual of $C_0(G)$. For $\mu\in V(G)$ and $s\in G$ let μ_s be the measure in $V(G)$ defined by $\mu_s(E)=\mu(Es^{-1})$ for each Borel set E. A measure μ is said to have separable orbit if there is a countable subset $C\subseteq G$ with the property that $s\in G$ and $\epsilon>0$ there exists a $c\in C$ such that $|\mu_s-\mu_c|(G)=\|\mu_s-\mu_c\|<\epsilon$.

In Theorem 2 of [2, p. 570], it was proved that a measure with separable orbit is absolutely continuous if G is second countable. We shall give a proof of this result without the second countability of G. When G is not σ-compact, $\mu\in V(G)$ has separable orbit if and only if G contains an open-closed subgroup H whose index is at most countable and μ on each coset of H is some multiple of the Haar measure on that set.

We begin with the following lemma which has been proved in [2, p. 570] under the condition that G is second countable. We give a different argument without this restriction.

LEMMA. *For every μ in $V(G)$ which has separable orbit,* $\lim_{s\to t}\|\mu_s-\mu_t\|=0$. *Thus if μ is in $M(G)$, the function $s\to\mu_s$ is continuous.*

PROOF. For each integer $m>0$,

$$G=\bigcup_{c\in C}\{x\in G\colon \|\mu_x-\mu_c\|\leqq 1/m\}.$$

Let x_α be in $S_c=\{x\colon\|\mu_x-\mu_c\|\leqq 1/m\}$ and converge to x. Since the function $f(x_\alpha s)$ converges to $f(s)$ uniformly with every f in $C_{00}(G)$,

$$\lim\int f(s)d\mu_{x_\alpha}(s)=\lim\int f(sx_\alpha)d\mu(s)=\int f(s)d\mu_x(s).$$

Thus $|\int f(s)d(\mu_x-\mu_c)|=|\lim\int f(s)d(\mu_{x\alpha}-\mu_c)|\leqq\lim\sup\|\mu_{x_\alpha}-\mu_c\|\,\|f\|_\infty\leqq\|f\|_\infty/m$ for all $f\in C_{00}(G)$. So $\|\mu_x-\mu_c\|\leqq 1/m$, and S_c is closed. As

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G is locally compact Hausdorff, it is of second category and there is a c_m so that $S_m = \{x: \|\mu_x - \mu_{c_m}\| \leqq 1/m\}$ has interior. Then there exists a neighborhood U_m of e, the identity of G, such that $x_m U_m \subseteq S_m$. Now for each $s \in U_m$, we have

$$\|\mu_s - \mu\| = \|\mu_{sx_m} - \mu_{x_m}\| \leqq \|\mu_{sx_m} - \mu_{c_m}\| + \|\mu_{c_m} - \mu_{x_m}\| \leqq 2/m.$$

Thus $\lim_{s\to e}\|\mu_s - \mu\| = 0$.

THEOREM 1. *Every measure μ in $V(G)$ which has a separable orbit is absolutely continuous.*

PROOF. For every compact set K in G we have

$$|\mu_s - \mu_t|(K) \leqq \|\mu_s - \mu_t\|.$$

Hence $\mu_t(K)$ is continuous as a function of t and μ is absolutely continuous [**3**, p. 230].

If G is not σ-compact, then it has an open-closed subgroup K such that G/K is uncountable [**1**, p. 34]. Every $\mu \in M(G)$ is supported on a countable union of cosets of K. So the orbit of μ cannot be separable unless $\mu = 0$. We wish to thank the referee for the following:

THEOREM 2. *Suppose G is not σ-compact. Then $\mu \in V(G)$ has separable orbit if and only if there is an open-closed subgroup H of G whose index in G is at most countable, and such that the restriction of μ to each coset of H is a multiple of the Haar measure on that set. In particular, if μ has separable orbit, then its orbit is at most countable.*

PROOF. Let H be the set $\{s \in G: \|\mu_s - \mu\| < \infty\}$. For any $s, t \in H$, $\|\mu_{st^{-1}} - \mu\| \leqq \|\mu_s - \mu_t\| \leqq \|\mu_s - \mu\| + \|\mu_t - \mu\| < \infty$. Hence H is a subgroup. By the Lemma, H is open and thus closed. Since the orbit of μ is separable, H has at most countable index in G. Moreover, H is not σ-compact. For all $t \in H$, $\mu_t - \mu \in M(G)$ has a separable orbit. So $\mu_t - \mu = 0$ and μ is invariant under translation by elements in H. Thus μ on coset of H must be a multiple of Haar measure on that set. The converse is clear.

THEOREM 3. *If G is σ-compact, and $\mu \in V(G)$ is such that $\lim_{s\to e}\|\mu_s - \mu\| = 0$, then μ has separable orbit. In particular, $\mu \in M(G)$ is absolutely continuous if and only if it has separable orbit.*

PROOF. Again let H be the set $\{s \in G: \|\mu_s - \mu\| < \infty\}$. As in the proof of Theorem 2, H is an open-closed subgroup. Since G is σ-compact, it has the Lindelöf property. The index of H in G is at most countable, Now the map $s \to \mu_s - \mu$ on H is continous in the total variation norm. So the set $\{\mu_s - \mu: s \in H\}$ is σ-compact and thus separable. It follows that the orbit of μ under translation by elements of H is separable. Since the index of H in G is at most countable, the orbit of μ is separable.

Bibliography

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