THE STRONG-BOUNDED TOPOLOGY ON GROUPS OF AUTOMORPHISMS OF A VON NEUMANN ALGEBRA

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0. Introduction. Let $R$ be a von Neumann algebra on the Hilbert space $H$, $G$ a topological group, and $a \mapsto \varphi(a)$ a representation of $G$ as a group of $*$-automorphisms of $R$. Recall that $a \mapsto \varphi(a)$ is continuous in the strong-bounded topology if and only if

$$\sup_{T \in R, \|T\| \leq 1} \| [\varphi(a)(T) - T]x \| \to 0 \quad (a \to e)$$

for all $x \in H$. The purpose of this note is to show that for certain von Neumann algebras $R$ and certain groups of $*$-automorphisms $\varphi(a)$, the continuity of $\varphi(a)$ in the strong-bounded topology is a very restrictive condition. For example, if $R$ is abelian and $s \mapsto \varphi(s)$ is a one-parameter group of $*$-automorphisms of $R$, continuous in the strong-bounded topology, then $\varphi(s)$ is the identity automorphism for all $s$. If $R$ is either a $I_\infty$ factor or a $III_\infty$ factor (and $H$ is separable), then a strong-bounded continuous one-parameter group of inner automorphisms must be uniformly continuous. Hence, strong-bounded continuous groups of automorphisms are probably not useful in quantum field theory, since the corresponding Hamiltonian operators have bounded spectrum.

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The notation and terminology of this note is that of Dixmier [1].

1. The results. We make two preliminary remarks before going on to the main theorems.

First, note that if $x \in H$ is cyclic for $R'$ and

$$\sup_{T \in R, \|T\| \leq 1} \| [\varphi(a)(T) - T]x \| \to 0 \quad (a \to e),$$

then $a \to \varphi(a)$ is continuous in the strong-bounded topology.

The second remark is contained in the following lemma.

**Lemma 1.1.** Let $R$ be a von Neumann algebra on $H$, $G$ a topological group, and $a \to \varphi(a)$ a representation of $G$ as a group of $*$-automorphisms

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of $R$, continuous in the strong-bounded topology. Let $S$ be a von Neumann algebra and $\phi: R \to S$ a $*$-isomorphism. Then $a \to \phi(a) \circ \phi^{-1}$ is a representation of $G$ as a group of $*$-automorphisms of $S$, continuous in the strong-bounded topology.

**Proof.** By Dixmier [1, Théorème 3, p. 58], it suffices to consider three special cases: (1) $\phi$ is an ampliation $\phi: T \to T \otimes I$, where $I$ is the identity on some Hilbert space $K$; (2) $\phi$ is an induction, $\phi: T \to T_E$, where $E$ is a projection in $R'$ with central support equal to $I$; (3) $\phi$ is a spatial isomorphism. Cases (2) and (3) are easy and are left to the reader. Case (1) may be reformulated as follows. There exists some index set $A$ such that $H \otimes K = \sum_{\alpha \in A} \oplus H_\alpha$, where for all $\alpha \in H_\alpha$ is a copy of $H$. $\phi$ then has the form $\phi: T \in R \to \sum_{\alpha \in A} \oplus T_\alpha \in R \otimes I$, where $T_\alpha = T$ for all $\alpha$. Given $x \in H \otimes K$, choose indices $\alpha_1, \ldots, \alpha_n$ such that $x_{\alpha_1} \otimes \cdots \otimes x_{\alpha_n}$ is as close in norm to $x$ as desired. Here $x_\beta$ is the $\beta$th component of $x$. Then choose $\delta$ so that

$$\sup_{|s| \leq \delta, \|T\| \leq 1, 1 \leq i \leq n} \| \phi(s)(T) - T \| x_{\alpha_i}$$

is as small as desired. Simple inequalities complete the proof. Q.E.D.

**Theorem 1.2.** Suppose $R$ is an abelian von Neumann algebra and $s \to \phi(s)$ is a one-parameter group of $*$-automorphisms of $R$, continuous in the strong bounded topology. Then $\phi(s)$ is the identity automorphism for all $s$.

Before proving Theorem 1.2, we make a few preliminary remarks and prove a preliminary lemma.

Let $\phi$ be a $*$-automorphism of $R$. Recall that $\phi$ is freely acting on $R$ if, given a nonzero projection $P$ in $R$, there exists a nonzero projection $Q$ in $R$ such that $Q \leq P$ and $Q \perp \phi(Q)$. In general, there exist projections $P$ and $Q$, fixed under $\phi$, such that $\phi$ is the identity on $R_Q$ and is freely acting on $R_P$. For each $\phi(s)$, let $P_s$ and $Q_s$ be the corresponding $P$ and $Q$. Note that $P_s = P_{-s}$ and $Q_s = Q_{-s}$. Theorem 1.2 will be proved if we can show $P_s = 0$ for all $s$.

Every abelian von Neumann algebra is $*$-isomorphic to a maximal abelian von Neumann algebra. Hence, by Lemma 1.1 it suffices to consider the case in which $R$ is maximal abelian. In this case the spectral theorem states that $R$ is unitarily equivalent to the multiplication algebra of some measure space $(M, \mu)$. From now on we assume that $R$ is such a multiplication algebra.

Next, a lemma needed in the proof of Theorem 1.2.
Lemma 1.3. Let \((\Omega, \nu)\) be a measure space, \(f \in L^2(\Omega, \nu)\), and \(\varphi\) a freely acting \(*\)-automorphism of \(L^\infty(\Omega, \nu)\). Then there exists a projection \(P \in L^\infty(\Omega, \nu)\) such that \(P \perp \varphi(P)\) and \(\|Pf\|_2 \geq \frac{1}{2}\|f\|_2\).

Proof. By Zorn's Lemma, choose a maximal projection \(P\) such that \(P \perp \varphi(P)\). \(P\) is nonzero since \(\varphi\) is freely acting on \(L^\infty(\Omega, \nu)\). Let \(Q = P + \varphi(P)\). Claim \(\varphi(I - Q) \leq P\). To show this it suffices to show that \(\varphi(I - Q) \leq Q\). If \(R = \varphi(I - Q) \cdot (I - Q) \neq 0\), then \(\varphi^{-1}(R) \leq I - Q \perp Q\) and \(R \leq I - Q \perp Q\). Since \(\varphi\) is freely acting, there exists a nonzero projection \(S \leq \varphi^{-1}(R)\) such that \(S \perp \varphi(S)\). But \(S \leq \varphi^{-1}(R) \perp Q\) and \(\varphi(S) \leq R \perp Q\). This contradicts the maximality of \(P\). Hence, \(\varphi(I - Q) \leq P\). At least one of \(\|Pf\|_2\), \(\|\varphi(P)f\|_2\), and \(\|I - Qf\|_2\) is greater than or equal to \(\frac{1}{2}\|f\|_2\). But \(I - Q \perp \varphi(I - Q)\), \(P \perp \varphi(P)\), and \(\varphi(P) \perp \varphi^2(P)\). The result follows. Q.E.D.

Easy examples show that the constant \(\frac{1}{2}\) in the above lemma is best possible.

Proof of Theorem 1.2. Let \(s\) be a positive real number, \(Q\) is a projection in \(L^\infty(M, \mu)\) corresponding to a measurable subset of \(M\) of finite measure, and \(\chi_Q\) the characteristic function of \(Q\). Now \((P_s, \mu)\) is a measure space and \(\varphi(s)\) is a freely acting \(*\)-automorphism of \(L^\infty(P_s, \mu)\). By Lemma 1.3 there exists a projection \(R \in L^\infty(P_s, \mu)\) such that

\[
\left[ \int_{P_s} R\chi_Q d\mu \right]^{\frac{1}{2}} \geq \frac{1}{3} \left[ \int_{P_s} \chi_Q d\mu \right]^{\frac{1}{2}}
\]

and \(R \perp \varphi(s)(R)\).

\[
\| [\varphi(s)(R) - R] \chi_Q \|_2^2 = \| P_s [\varphi(s)(R) - R] \chi_Q \|_2^2 = \left[ \int_{P_s} | \varphi(s)(R) - R |^2 \chi_Q d\mu \right]^{\frac{1}{2}}
\]

\[
= \left[ \int_{P_s} [\varphi(s)(R) \chi_Q + R \chi_Q] d\mu \right]^{\frac{1}{2}} \geq \left[ \int_{P_s} R \chi_Q d\mu \right]^{\frac{1}{2}}
\]

\[
\geq \frac{1}{3} \left[ \int_{P_s} \chi_Q d\mu \right]^{\frac{1}{2}} = \frac{1}{3} \mu(P_s Q)^{\frac{1}{2}}.
\]

Hence,

\[
9 \sup_{|r| \leq n, T \in R, \|T\|_1 \leq 1} \| [\varphi(r)(T) - T] \chi_Q \|_2^2 \geq \mu(P_s Q).
\]

If \(\varphi\) is a \(*\)-automorphism of \(R\) and \(\varphi\) leaves a projection \(R\) absolutely fixed, then \(\varphi^n\) leaves \(R\) absolutely fixed \((n \geq 1)\). Hence, \(Q_s \leq Q_s\), and therefore \(P_s \geq P_s\). From this it follows that
\[ \mu(P_sQ) \leq \mu(P_s/nQ) \]
\[ \leq 9 \sup_{|r| \leq s/n, T \in \mathcal{R}, \|T\| \leq 1} \| [\varphi(r)(T) - T]XQ \|_2 \to 0 \quad (n \uparrow + \infty). \]

Hence, \( P_s \) is orthogonal to \( Q \). Since \( Q \)’s corresponding to measurable sets of finite measure generate \( \mathcal{R} \), \( P_s = 0 \) for all \( s \).

As noted above, this proves Theorem 1.2. Q.E.D.

**Theorem 1.4.** Suppose \( \mathcal{R} \) is either: (1) a \( I_\infty \) factor; or (2) \( \mathcal{H} \) is separable and \( \mathcal{R} \) is a \( III_\infty \) factor. Let \( s \to U(s) \) be a strongly continuous one-parameter unitary group in \( \mathcal{R} \). Let \( \varphi(s)(T) = U(s)TU(-s) \quad (T \in \mathcal{R}) \). Suppose \( \varphi(s) \) is continuous in the strong-bounded topology. Then \( s \to U(s) \) is uniformly continuous.

**Proof.** Assume \( \mathcal{R} \) is a \( I_\infty \) factor. Let \( U(s) = e^{iA} \). \( U(s) \) is uniformly continuous if and only if \( A \) is bounded. If \( U(s) \) is not uniformly continuous, assume that the spectrum of \( A \) is unbounded on the positive real axis (otherwise consider the one-parameter unitary group \( V(s) = U(-s) \)). Let \( U(s) = \int_{-\infty}^{\infty} e^{i\lambda s}dE(\lambda) \). Then there exist integers \( n_0 < n_1 < \cdots \uparrow + \infty \) such that \( E([n_k, n_k+1)) \neq \emptyset \quad (k \geq 0) \). Choose a minimal projection \( P_0 \leq E([n_0, n_0+1)) \) and \( x \in P_0(\mathcal{H}), \quad \|x\| = 1 \).

Choose partial isometrics \( U_k \quad (k \geq 1) \) such that \( U_k^*U_k = P_0 \) and \( P_k = U_kU_k^* \leq E([n_k, n_k+1)) \). Now since multiplication on the right by \( U(s) \) carries the unit ball of \( \mathcal{R} \) onto itself

\[
\sup_{T \in \mathcal{R}, \|T\| \leq 1} \| [U(s)TU(-s) - T]x \| = \sup_{T \in \mathcal{R}, \|T\| \leq 1} \| [U(s)T - TU(s)]x \|.
\]

\[
\sup_{T \in \mathcal{R}, \|T\| \leq 1} \| [U(s)U_k - U_kU(s)]x \| 
\geq \| \exp(in_k s)E([n_k, n_k+1))U_kx - \exp(in_0 s)U_kE([n_0, n_0+1])x \| 
- \| U(s)U_kx - \exp(in_k s)E([n_k, n_k+1))U_kx \| 
- \| U_k \exp(in_0 s)E([n_0, n_0+1])x - U_kU(s)x \| 
\geq \| \exp(in_k s) - \exp(in_0 s) \| 
\geq \sup_{\lambda \in [n_k, n_k+1]} \| \exp(i\lambda s) - \exp(in_k s) \| 
\geq 1 \| \exp(i(n_k - n_0)s) - 1 \| - 2 \sup_{\lambda \in [0, 1]} \| \exp(i\lambda s) - 1 \|.
\]

Now \( n_k - n_0 \uparrow + \infty \) as \( k \uparrow + \infty \). Hence,

\[
\lim \sup_{s \to 0, T \in \mathcal{R}, \|T\| \leq 1} \| [U(s)TU(-s) - T]x \| = 2.
\]

**Contradiction.** Hence, \( U(s) \) must be uniformly continuous.
Now assume that $R$ is a $III_\infty$ factor on a separable Hilbert space. The proof of the theorem for this case follows the proof of the $I_\infty$ case almost verbatim. The only twist is the use of the fact that two non-zero projections in a $III_\infty$ factor on a separable Hilbert space are equivalent in the Murray-von Neumann sense. The details are left to the reader. Q.E.D.

We remark that if $R$ is a $I_\infty$ factor on a separable Hilbert space, then the following apparent strengthening of Theorem 1.4 is true. If $\varphi(s)$ is an arbitrary one-parameter group of $*$-automorphisms of $R$, continuous in the strong-bounded topology, then $s \mapsto \varphi(s)$ is continuous in the norm topology. A sketch of the proof of this follows. $\varphi(s)$ is inner for each $s$ since $R$ is a $I_\infty$ factor. $\varphi(s)$ continuous in the strong-bounded topology implies that $\varphi(s)$ is continuous in the weak-bounded topology. Since $H$ is separable, a result of Kadison ([2, Theorem 4.13, p. 195]) now shows that there exists a strongly continuous one-parameter unitary group $s \mapsto U(s)$ in $R$ such that $\varphi(s)(T) = U(s)TU(-s)$. Theorem 1.4 shows $s \mapsto U(s)$ is uniformly continuous. Easy estimates now imply that $s \mapsto \varphi(s)$ is continuous in the norm topology.

It is unknown to the author whether or not an analogue of Theorem 1.4 holds for $R$ a $II_\infty$ factor. The $II_1$ case is handled by the next theorem.

**THEOREM 1.5.** Let $R$ be a $II_1$ von Neumann algebra and $U(s)$ a strongly continuous one-parameter unitary group in $R$. Let $\varphi(s)(T) = U(s)TU(-s)$. Then $s \mapsto \varphi(s)$ is continuous in the strong-bounded topology.

**Proof.** There exists some index set $B$ such that $R = \bigoplus_{\beta \in B} R_\beta$, where each $R_\beta$ has a faithful finite trace. Note that each $\varphi(s)$ leaves $\text{Cent } R$ fixed and hence $\varphi(s) = \bigoplus_{\beta \in B} \varphi_\beta(s)$, where $\varphi_\beta(s)$ is a one-parameter group of $*$-automorphisms of $R_\beta$. An argument like that used in the proof of Lemma 1.1 shows that $\varphi(s)$ is continuous in the strong-bounded topology if each $\varphi_\beta(s)$ is. Hence, we may assume $R$ has a faithful finite trace. By Lemma 1.1, we may assume $R$ has a trace vector $x$. By the remarks preceding Lemma 1.1, it suffices to show that

$$\sup_{T \in \mathcal{R}, \|T\|_2 \leq 1} \|[\varphi(s)(T) - T]x\| \to 0 \quad (s \to 0);$$

i.e.,

$$\sup_{T \in \mathcal{R}, \|T\|_2 \leq 1} \|\varphi(s)(T) - T\|_2$$

$$= \|U(s)T - TU(s)\|_2 \to 0 \quad (s \to 0)$$

(here $\|\cdot\|_2$ denotes the trace norm).
Let $P$ be a "large" spectral projection of

$$U(s) = \int_{-\infty}^{+\infty} e^{i\lambda s} dE(\lambda) \quad \text{(say } P = E([-n, n]) \text{ for large } n)$$

such that $\|I-P\|_2 < \epsilon$. Let $T \in \mathcal{R}$, $\|T\| \leq 1$. Then

$$\|\varphi(s)(T) - T\|_2 \leq \|PU(s)PTP - PTU(s)P\|_2 + 6\epsilon$$

$$\leq 2\|PU(s)P - P\|_\infty + 6\epsilon.$$ 

$\|PU(s)P - P\|_\infty \to 0$ as $s \to 0$ since $PU(s)P$ is a unitary operator with bounded spectrum on $P(H)$. Hence, $\|\varphi(s)(T) - T\|_2 \to 0$ as $s \to 0$.

Q.E.D.

We note that Theorem 1.5 is not true for an arbitrary one-parameter group $\varphi(s)$ of $*$-automorphisms of a $II_1$ factor. For example, for $n \geq 1$, let $R_n$ be the algebra of all $2 \times 2$ matrices, $\tau_n$ the normalized trace on $R_n$, and $\varphi_n(s)$ the $*$-automorphism of $R_n$ given by the unitary $e^{ins}0e^{-ins}$. Further, let $S$ be the $C^*$-tensor product $\otimes_{n \geq 1} R_n$, $\tau = \otimes_{n \geq 1} \tau_n$, 

$$\begin{pmatrix} e^{ins} & 0 \\ 0 & e^{-ins} \end{pmatrix}$$

and $\varphi(s) = \otimes_{n \geq 1} \varphi_n(s)$. If $\Pi_r$ is the cyclic representation of $S$ with cyclic vector $\xi$, on the Hilbert space $H$, corresponding to the state $\tau$, $\Pi_r$ is faithful, and $\Pi_r(S)'s$, the strong closure of $\Pi_r(S)$, is a $II_1$ factor. Since $\varphi(s)$ leaves $\tau$ invariant, there exists a strongly continuous one-parameter unitary group $U(s)$ on $H$, such that

$$U(s)\Pi(T)U(-s) = \Pi(\varphi(s)(T)) \quad (T \in S).$$

Easy computations show that

$$\lim_{s \to 0, T \in \Pi_r(s), \|T\| \leq 1} \| [U(s)TU(-s) - T]\xi \| = 2.$$

**Bibliography**


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