THE SET OF CURVILINEAR CONVERGENCE OF A CONTINUOUS FUNCTION DEFINED IN THE INTERIOR OF A CUBE

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Let Ω be an open connected set in a finite-dimensional Euclidean space, and let f be a function mapping Ω into another finite-dimensional Euclidean space. We define the *set of curvilinear convergence of* f to be

$\{p \in$ boundary of Ω: there exists a simple arc γ with one endpoint at p such that $\gamma - \{p\} \subseteq \Omega$ and $f(v)$ converges to a finite limit as $v \to p$ along $\gamma\}$.

J. E. McMillan [6] has shown that if Ω is an open disk in the plane and if f is continuous in Ω, then the set of curvilinear convergence of f is of type $F_{\sigma\delta}$. In this paper we prove that there exists a bounded continuous complex-valued function f, defined in the interior of a three-dimensional cube, such that the set of curvilinear convergence of f is not a Borel set. Thus McMillan's theorem does not generalize to three dimensions. However, the following question remains open.

PROBLEM. *Does there exist a continuous real-valued function f, defined in the interior of a three-dimensional cube, such that the set of curvilinear convergence of f is not a Borel set?*

Let

R be the set of real numbers
$R^n = n$-dimensional Euclidean space
$Q = \{(x, y) \in R^2: 0 < y \leq 1 \text{ and } -1 \leq x \leq 1\}$
$K = \{(x, y, z) \in R^3: 0 < y \leq 1, -1 \leq x \leq 1, \text{ and } -1 \leq z \leq 1\}$
$Q^\circ = $ interior of Q
$K^\circ = $ interior of K.

Let Ω again represent an open connected subset of R^n. If $f: \Omega \to R^m$ is a function, we shall say that $a \in R^m$ is an *asymptotic value* of f iff there exists a continuous function $v: [0,1) \to \Omega$ such that $\mathrm{dist}(v(t), R^n - \Omega) \to 0$ and $f(v(t)) \to a$ as $t \to 1$. (Note that a limit approached by f along a path which tends to ∞ may or may not be an asymptotic value by our definition.) We say that a is a *point asymptotic value* of f (at p) iff v can be chosen so that, as $t \to 1$, $v(t)$ approaches

a point $p \in R^n - \Omega$. Because of the result of [8], the set of curvilinear convergence of f is

$$\{ p \in R^n - \Omega : f \text{ has a point asymptotic value at } p \}.$$

LEMMA. *There exists a continuous complex-valued function s defined in*

$$\{ (x, y) \in R^2 : y > 0 \},$$

with $|s(x, y)| \leqq 1$ for all x and y, such that s has the following property. Let E be the set of all asymptotic values of s that are real and lie in the interval $(-1, 1)$. Then E is equal to the set of all point asymptotic values of s that are real and lie in $(-1, 1)$, and E is not a Borel set.

PROOF. Let A be an analytic subset of R that is not a Borel set. (This exists [7, p. 254].) We see from the paper of Kierst [4] that there exists a holomorphic functlon h defined in $\{ z : z$ is a complex number and $|z| < 1 \}$ such that h omits the three values $-i, i, \infty$ and $A \cup \{ -i, i \}$ is the set of all (finite) asymptotic values of h. The function h is then normal [5, p. 53], so, as pointed out by McMillan [6, p. 311], it follows from Theorem 1 of [2] that $A \cup \{ -i, i \}$ is just the set of all (finite) point asymptotic values of h. We now obtain the desired function by setting

$$s(x, y) = \frac{h((1 - y)e^{ix})}{1 + |h((1 - y)e^{ix})|} \qquad (0 < y \leqq 1),$$

$$s(x, y) = \frac{h(0)}{1 + |h(0)|} \qquad (y \geqq 1).$$

REMARK. Since the theorem we want to prove has nothing to do with meromorphic functions, it is unfortunate that the proof of the lemma depends on the theory of meromorphic functions. This can be avoided. The lemma can be proved by using [7, Theorem 113, p. 216], [1, Theorem 2, p. 179], and the methods of [3], but this involves a messy construction, so we omit the details.

THEOREM. *There exists a bounded continuous complex-valued function f defined in K° such that the set of curvilinear convergence of f is not a Borel set.*

PROOF. Let s and E be as described in the lemma, and set $g(x, y) = s(x/y, y)$ for $(x, y) \in Q$. The reader can verify that E equals the set of all real point asymptotic values of g at the point $(0, 0)$ which lie in the interval $(-1, 1)$. For each $t \in (0, 1]$, define

$$d_0(t) = \sup\{\delta \in (0, 1]: ((x, y) \in Q, (x', y') \in Q, y \geqq t, y' \geqq t,$$
$$\text{and } | (x, y) - (x', y')| < \delta) \text{ implies } | g(x, y) - g(x', y')| \leqq t\},$$
$$d(t) = \min\{\tfrac{1}{2}d_0(\tfrac{1}{2}t), \tfrac{1}{4}t\}.$$

By statement (C) of [3], there exists a continuous complex-valued function k defined in Q, with $|k(x, y)| \leqq 2^{1/2}$ for all $(x, y) \in Q$, such that for each $a \in (0, 1]$ and for each arc

$$\gamma \subseteq \{(x, y): -1 \leqq x \leqq 1 \text{ and } 0 < y \leqq a\},$$

(diameter $\gamma) \geqq d(a)$ implies (diameter $k(\gamma)) \geqq 2$.

Let f be the function with domain K° defined by $f(x, y, z) = (g(x, y) - z)k(x, y)$. We note that the following inequality holds for any three points (x, y, z), (x', y', z'), (x'', y'', z'') in K°:

$$|f(x', y', z') - f(x'', y'', z'')|$$
$$= |(g(x', y') - z')k(x', v') - (g(x, y) - z)k(x', y')$$
$$+ (g(x, y) - z)k(x', y') - (g(x, y) - z)k(x'', y'')$$
$$+ (g(x, y) - z)k(x'', y'') - (g(x'', y'') - z'')k(x'', y'')|$$
$$\geqq |g(x, y) - z||k(x', y') - k(x'', y'')|$$

(1)
$$- |k(x', y')||g(x', y') - z' - g(x, y) + z|$$
$$- |k(x'', y'')||g(x, y) - z - g(x'', y'') + z''|$$
$$\geqq |g(x, y) - z||k(x', y') - k(x'', y'')|$$
$$- 2|g(x', y') - g(x, y)| - 2|g(x, y) - g(x'', y'')|$$
$$- 2|z - z'| - 2|z'' - z|.$$

Let $L = \{(0, 0, z): -1 < z < 1\}$, and let Γ be the set of curvilinear convergence of f. We wish to show that $\Gamma \cap L = \{(0, 0, z): z \in E\}$. Suppose $b \in E$. Then there is an arc γ with one endpoint at $(0, 0)$ such that $\gamma - \{(0, 0)\} \subseteq Q^\circ$ and g approaches b along γ. Let

$$\gamma' = \{(x, y, b): (x, y) \in \gamma\}.$$

Then $g(x, y) - z \to 0$ as $(x, y, z) \to (0, 0, b)$ along γ'. Thus, since k is bounded, $f(x, y, z) \to 0$ along γ', so $(0, 0, b) \in \Gamma \cap L$.

Now let us assume, conversely, that $(0, 0, b) \in \Gamma \cap L$ and deduce that $b \in E$. Let γ' be an arc with one endpoint at $(0, 0, b)$ such that $\gamma' - \{(0, 0, b)\} \subseteq K^\circ$ and f approaches a limit along γ'. Let

$$\gamma = \{(x, y) \in R^2: (x, y, z) \in \gamma' \text{ for some } z\}.$$

Then γ is a (not necessarily simple) arc with one endpoint at $(0, 0)$ and $\gamma - \{(0, 0)\} \subseteq Q^\circ$. I assert that $g(x, y) - z$ approaches 0 along γ'.

Assume this is false. Then there exists $\epsilon > 0$ and there exists a sequence of points $\{(x_n, y_n, z_n)\}_{n=1}^{\infty}$ in $\gamma' - \{(0, 0, b)\}$ such that

$$(x_n, y_n, z_n) \to (0, 0, b) \qquad \text{as } n \to \infty$$

and $|g(x_n, y_n) - z_n| \geqq \epsilon$ for all n. Let $\delta > 0$ be chosen so that whenever $(u, v, w) \in \gamma'$, $(x, y, z) \in \gamma'$, and $v, y \leqq \delta$, then $|w - z| < \frac{1}{8}\epsilon$. Let N be chosen so that $n \geqq N$ implies $y_n < \min\{3\epsilon/32, 3\delta/4, 3/4\}$.

For the present, let n be a fixed integer greater than N. Set $a = 4y_n/3$. There exists an arc γ^* contained in

$$\gamma \cap \{(x, y) \in R^2 : |(x, y) - (x_n, y_n)| \leqq d(a)\}$$

joining (x_n, y_n) to a point on the circle of radius $d(a)$ about (x_n, y_n). Clearly (diameter $\gamma^*) \geqq d(a)$, so (diameter $k(\gamma^*)) \geqq 2$. Choose points (x_n', y_n'), (x_n'', y_n'') in γ^* with $|k(x_n', y_n') - k(x_n'', y_n'')| \geqq 2$. Choose z_n', z_n'' so that (x_n', y_n', z_n') and (x_n'', y_n'', z_n'') are in γ'. It is easy to check that $\frac{1}{2}a \leqq y_n' < \delta$ and $\frac{1}{2}a \leqq y_n'' < \delta$, so

$$(2) \qquad |z_n - z_n'| < \tfrac{1}{8}\epsilon \quad \text{and} \quad |z_n'' - z_n| < \tfrac{1}{8}\epsilon.$$

Moreover, since $|(x_n', y_n') - (x_n, y_n)| \leqq d(a) \leqq \frac{1}{2}d_0(\frac{1}{2}a)$, we have

$$|g(x_n, y_n) - g(x_n', y_n')| \leqq \tfrac{1}{2}a < \tfrac{1}{8}\epsilon;$$

and similarly

$$|g(x_n'', y_n'') - g(x_n, y_n)| < \tfrac{1}{8}\epsilon.$$

Combining these inequalities with (1) and (2), we get

$$|f(x_n', y_n', z_n') - f(x_n'', y_n'', z_n'')| >$$
$$|g(x_n, y_n) - z_n||k(x_n', y_n') - k(x_n'', y_n'')| - \epsilon \geqq 2\epsilon - \epsilon = \epsilon.$$

But $y_n', y_n'' \leqq 4y_n/3$, so $(x_n', y_n', z_n') \to (0, 0, b)$ and $(x_n'', y_n'', z_n'') \to (0, 0, b)$ as $n \to \infty$; hence f cannot approach a limit along γ', which is a contradiction. We conclude that $g(x, y) - z \to 0$ as $(x, y, z) \to (0, 0, b)$ along γ'.

It follows immediately that $g(x, y) \to b$ along γ, so $b \in E$. We have now shown that

$$\Gamma \cap L = \{(0, 0, z) : z \in E\}.$$

Thus $\Gamma \cap L$ is not a Borel set. Hence Γ is not a Borel set; for if it were, then $\Gamma \cap L$ would also be a Borel set.

References

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