

# COMPACTIFICATION OF MAPPINGS

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**1. Introduction.** A mapping (continuous function)  $f: X \rightarrow f(X) = Y$  is *compact* if it is closed and point inverses are compact. (These mappings are often called *perfect*, or *proper*.) A compact mapping has the property that the inverse image of any compact set is compact. This property is sometimes taken as the definition of a compact mapping, although the two definitions are not equivalent for arbitrary range  $Y$ . Exact conditions for their equivalence are given in reference [6].

A *compactification* of  $f$  is a pair  $(X^*, f^*)$  where  $X^*$  is a Hausdorff space containing  $X$  as a dense subspace, and  $f^*$  is a compact mapping of  $X^*$  onto  $Y$  such that  $f^*|X = f$ . G. T. Whyburn introduced the notion of a mapping compactification in 1953 when he showed that every mapping of one locally compact Hausdorff space onto another is a partial mapping of a compact mapping on a Hausdorff space [4], [7]. He also noted in reference [7] that a compactification for any mapping from one completely regular space onto another can be obtained by restricting the extension  $\bar{f}: \beta X \rightarrow \beta Y$  to  $\bar{f}^{-1}(Y)$ .

In this paper we study some general properties of mapping compactifications and construct a class of so-called filter space compactifications of a mapping from a completely regular space onto a regular space. Each of these compactifications is associated in a natural way with a compactification of the domain of  $f$ . For  $Y$  locally compact, we show that the domain  $X$  is completely regular if and only if  $f$  has a compactification; and if  $X$  is completely regular, every compactification of  $f$  is a filter space compactification. We shall assume that all spaces are Hausdorff. Thus in this paper, regular spaces, completely regular spaces, etc., are Hausdorff.

**2. Mapping compactifications.** Let  $R$  denote the collection of all points  $p$  in  $Y$  having the property that there is a filter  $\mathbf{F}$  on  $X$  with no accumulation point such that  $f(\mathbf{F})$  converges to  $p$ . The following result then follows directly from Theorem 1, p. 101, of [1].

**PROPOSITION 2.1.** *The restriction of  $f$  to  $f^{-1}(Y - R)$  is compact, and if  $f$  is compact,  $R$  is empty.*

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In case  $X$  is regular, we need only consider *open filters*; that is, those which have a base consisting of open sets. Specifically, we have

**PROPOSITION 2.2.** *If  $X$  is regular,  $R$  consists of those points  $p$  in  $Y$  having the property that there is an open filter  $\mathbf{G}$  on  $X$  with no accumulation point which is such that  $f(\mathbf{G})$  converges to  $p$ .*

**PROOF.** Suppose  $p$  is in  $R$ ,  $\mathbf{F}$  has no accumulation point, and  $f(\mathbf{F})$  converges to  $p$ . Let  $\mathcal{V}$  be the collection of all closed neighborhoods in  $X$  which have a member of  $\mathbf{F}$  contained in their complement. The fact that  $\mathbf{F}$  has no accumulation point guarantees that each  $x$  in  $X$  has at least one neighborhood in  $\mathcal{V}$ .

The collection  $\alpha = \{X - V_\alpha \mid V_\alpha \in \mathcal{V}\}$  is a filter base. If  $\mathbf{A}$  denotes the filter generated by  $\alpha$ , then clearly  $\mathbf{G} = \{A \cap f^{-1}(U) \mid A \in \mathbf{A}, U \text{ is a neighborhood of } p\}$  is an open filter having no accumulation point, and  $f(\mathbf{G})$  converges to  $p$ .

We shall let  $S$  denote the singular set of  $f$ . ( $S$  is the collection of all  $p$  in  $Y$  such that in every neighborhood of  $p$  there is a compact set with a noncompact inverse image [2], [5].) It follows easily from the fact that  $f$  restricted to  $f^{-1}(Y - \text{cl}(R))$  is compact and the results in reference [6] that  $S \subset \text{cl}(R)$ , and that if  $Y$  is a  $k$ -space then  $R \subset S = \text{cl}(R)$ .

**THEOREM 2.3.** *If  $(X^*, f^*)$  is a compactification of  $f$ , then  $f^*(X^* - X) = R$ .*

**PROOF.** Let  $p$  be in  $R$  and suppose  $\mathbf{F}$  is a filter on  $X$  such that  $f(\mathbf{F})$  converges to  $p$  and  $\mathbf{F}$  has no accumulation point in  $X$ .  $f^*$  is compact, so there is an accumulation point  $x$  of  $\mathbf{F}$  in  $X^* - X$ , and from the continuity of  $f$  it follows that  $f^*(x) = p$ .

For  $x$  in  $X^* - X$ , let  $\mathbf{N}$  be the neighborhood filter of  $x$ . Since  $X$  is dense in  $X^*$ ,  $\mathbf{F} = \{N \cap X \mid N \in \mathbf{N}\}$  is a filter on  $X$  without an accumulation point in  $X$  and is such that  $f(\mathbf{F})$  converges to  $f^*(x)$ .

**COROLLARY.** *If  $X$  is locally compact,  $R$  is closed. If in addition,  $Y$  is a  $k$ -space, then  $R = S$ .*

**PROOF.** If  $X$  is locally compact,  $X$  is open in  $X^*$ . Thus  $f^*(X^* - X)$  is closed.

A relation  $\cong$  is defined on the collection of all compactifications of  $f$  by agreeing that  $(X_2, f_2) \cong (X_1, f_1)$  if there is a mapping  $h$  of  $X_2$  onto  $X_1$  which leaves the points of  $X$  fixed and is such that  $f_2 = f_1 h$ . If the mapping  $h$  can be taken to be a homeomorphism,  $(X_1, f_1)$  and  $(X_2, f_2)$  are said to be *equivalent*. It is easy to verify that the relation  $\cong$  is a partial order and that  $(X_1, f_1)$  and  $(X_2, f_2)$  are equivalent if and only if the relations  $(X_2, f_2) \cong (X_1, f_1) \cong (X_2, f_2)$  hold.

In case  $X$  and  $Y$  are locally compact, Whyburn's unified space of the mapping  $f$  provides a compactification  $(X_w, f_w)$  in which  $f_w| (X_w - X)$  is a homeomorphism onto  $R$  [4]. The next result shows that any compactification with this property precedes all others with respect to the partial order  $\cong$ . Thus all such compactifications of  $f$  are equivalent.

**THEOREM 2.4.** *Suppose the domain of  $f$  is locally compact. If  $(X_w, f_w)$  is a compactification of  $f$  such that  $f_w$  restricted to  $X_w - X$  is a homeomorphism, then  $(X^*, f^*) \cong (X_w, f_w)$  for any compactification  $(X^*, f^*)$  of  $f$ .*

**PROOF.** Define the function  $h: X^* \rightarrow X_w$  by setting  $h(x) = x$  for  $x$  in  $X$  and  $h(x) = [f_w^{-1}f^*(x)] \cap (X_w - X)$  for  $x$  in  $X^* - X$ . It is clear that  $h$  is a function, that  $f^* = f_w h$ , and that  $h$  is continuous at all points of  $X$ . To show that  $h$  is continuous at a point  $x$  in  $X^* - X$ , let  $\mathbf{N}$  be the neighborhood filter of  $x$ . Then  $f^*(\mathbf{N})$  converges to  $f^*(x)$ , and  $f_w$  is compact, so  $h(\mathbf{N})$  has at least one accumulation point  $z$ . From the fact that  $X$  is open in  $X_w$  and  $X^*$ , and  $h$  is the identity on  $X$ , it follows that  $z$  can not be in  $X$ , so  $h(\mathbf{N})$  has exactly one accumulation point  $z = h(x)$  in  $X_w - X$ .

Let  $W$  be a neighborhood of  $z$ . If every member of  $h(\mathbf{N})$  meets the complement of  $W$ , then  $\mathbf{G} = \{G \cap (X_w - W) \mid G \in h(\mathbf{N})\}$  is a filter on  $X_w$  with no accumulation point, and  $f_w(\mathbf{G})$  converges to  $y$ . This contradicts the fact that  $f_w$  is compact. Thus  $h(\mathbf{N})$  converges to  $z = h(x)$  and  $h$  is continuous.

**3. Filter space compactifications.** Consider a mapping  $f: X \rightarrow f(X) = Y$ , where  $X$  is completely regular and  $Y$  is regular. Let  $\bar{X}$  be a compactification (Hausdorff) of the space  $X$ . For  $y \in Y$ , let  $\bar{C}(y) \subset \bar{X}$  be the collection of all accumulation points in  $\bar{X}$  of the inverse image of the neighborhood filter of  $y$ . Note that  $f^{-1}(y) \subset \bar{C}(y)$  and  $f^{-1}(y) = \bar{C}(y)$  if and only if  $y \in Y - R$ .

With the above assumptions and terminology, we now have the following theorem.

**THEOREM 3.1.** *The mapping  $f$  has a compactification  $(X^*, f^*)$  with the property that for each  $y \in Y$ , there is a homeomorphism  $h_y$  of  $f^{*-1}(y)$  onto  $\bar{C}(y)$  which leaves the points of  $f^{-1}(y)$  fixed.*

We shall construct  $(X^*, f^*)$  and establish via a series of propositions that it is a compactification of  $f$  having the property in the theorem.

In constructing  $(X^*, f^*)$ , extensive use of the idea of a *filter space* as developed by F. J. Wagner [3] will be made. If  $\bar{X}$  is a collection of open filters on  $X$  which includes the neighborhood filter of every  $x$

in  $X$ , then  $\bar{X}$  can be made into a topological space by taking the collection of sets  $\bar{U} = \{F \in \bar{X} \mid U \in F, U \text{ is an open subset of } X\}$  as a base for open sets in  $\bar{X}$ . The space  $\bar{X}$  is known as a filter space over  $X$ . It is shown in reference [3] that the function  $e: X \rightarrow e(X) \subset \bar{X}$  defined by  $e(x) =$  the neighborhood filter of  $x$ , is a homeomorphism and  $e(X)$  is a dense subspace of  $\bar{X}$ .

If  $\bar{X}$  is a compactification of the space  $X$ , we may consider  $\bar{X}$  to be a filter space over  $X$  by associating with each point in  $\bar{X}$  the trace on  $X$  of its neighborhood filter (Theorem 2 of [3]). For each filter  $F$  in  $\bar{X}$ , consider  $\alpha_y = \{F \cap f^{-1}(W_y) \mid F \in F, W_y \text{ a neighborhood of } y\}$ . Each  $\alpha_y$  that does not contain the empty set is a base for a filter on  $X$ , which we shall denote by  $F_y$ . Note that  $F_y$  is defined if and only if  $F \cap f^{-1}(W_y) \neq \emptyset$  for every neighborhood  $W_y$  of  $y$  and every  $F \in F$ .

To avoid confusion, we shall throughout use the terminology " $F$  is a filter on  $X$ " when  $F$  is a collection of subsets of  $X$ , and " $F$  is a filter in  $X$ " when  $F$  is a member of  $X$ .

PROPOSITION 3.2. *For  $F$  and  $G$  in  $\bar{X}$ ,  $F_y = G_z$  if and only if  $F = G$ , and  $y = z$ .*

PROOF. The "if" part is trivial. Assume  $F_y = G_z$ . The inverse image of every neighborhood of  $y$  is a member of  $F_y$  and the inverse image of every neighborhood of  $z$  is a member of  $G_z$ , so it follows from the fact that  $Y$  is Hausdorff that  $y = z$ . That  $F = G$  follows from the fact that  $F \subset F_y$ ,  $G \subset G_z$ , and the Hausdorff property of  $\bar{X}$ .

It is clear that each  $F_y$  is an open filter and that for  $F = N_x$ , the neighborhood filter of  $x$ , then  $F_y = N_x$ , where  $y = f(x)$ . Let  $X^*$  denote the filter space consisting of all such  $F_y$ . Define the function  $f^*$  from  $X^*$  onto  $Y$  by setting  $f^*(F_y) = y$ . Then  $f^*|_X = f$  and  $X$  is dense in  $X^*$ .

PROPOSITION 3.3.  *$f^*$  is continuous.*

PROOF. Let  $R$  be a neighborhood of  $y = f^*(F_y)$  and choose  $W$  to be an open neighborhood of  $y$  such that  $\text{cl}(W) \subset R$ . Now  $U = f^{-1}(W)$  is a member of  $F_y$ , so  $U^* = \{G_z \in X^* \mid U \in G_z\}$  is a neighborhood of  $F_y$ . For any  $G_z \in U^*$ ,  $f^{-1}(W) \in G_z$ , so  $z = f^*(G_z) \in \text{cl}(W) \subset R$ .

PROPOSITION 3.4.  *$X^*$  is regular.*

PROOF. Let  $F_y$  and  $G_z$  be two distinct points in  $X^*$ . If  $y \neq z$ , the existence of disjoint neighborhoods of the two points follows from the continuity of  $f^*$  and the Hausdorff property of  $Y$ . If  $F = G$ , there are disjoint open sets  $U$  and  $V$  in  $X$  such that  $U \in F$  and  $V \in G$  since  $\bar{X}$  is Hausdorff. Thus  $U^*$  and  $V^*$  are disjoint neighborhoods of  $F_y$  and  $G_z$ , respectively.

Next, let  $F_y \in U^*$ , and choose an open  $F \in \mathbf{F}$  and a neighborhood  $W_y$  of  $y$  so that  $F \cap f^{-1}(W_y) \subset U$ . Let  $M$  be an open neighborhood of  $y$  such that  $\text{cl}(M) \subset W_y$ . The space  $\tilde{X}$  is regular, so choose an open  $V \subset X$  so that  $V \in \mathbf{F}$  and  $\text{cl}(\tilde{V}) \subset \tilde{F}$ , where  $\tilde{V}$  denotes the collection of all elements in  $\tilde{X}$  that have  $V$  as a member. Let  $R = V \cap f^{-1}(M)$ . We shall show that  $R^*$  is a neighborhood of  $F_y$  such that  $\text{cl}(R^*) \subset U^*$ .

First note that since  $V \in \mathbf{F}$ , we have  $V \cap f^{-1}(M) = R \in F_y$ , so  $R^*$  is indeed a neighborhood of  $F_y$ . If  $G_z \in \text{cl}(R^*)$ , every open set in  $G_z$  meets  $R$ , so  $z \in \text{cl}(M) \subset W_y$ . Let  $W_z$  be a neighborhood of  $z$  such that  $W_z \subset W_y$ . For every  $G \in \mathbf{G}$ , we have  $[G \cap f^{-1}(W_z)] \cap R \neq \emptyset$ , and hence  $G \cap V \neq \emptyset$  for every  $G$  in  $\mathbf{G}$ . This means that  $G \in \text{cl}(\tilde{V}) \subset \tilde{F}$ , or  $F \in \mathbf{G}$ . Hence  $F \cap f^{-1}(W_z) \in G_z$ , and  $U \in G_z$  since  $F \cap f^{-1}(W_z) \subset U$ .

PROPOSITION 3.5. *If  $\mathfrak{F}$  is an open filter on  $X^*$  such that  $f^*(\mathfrak{F})$  converges to some  $y$  in  $Y$ , then  $\mathfrak{F}$  has an accumulation point.*

PROOF.  $\mathfrak{F} \cap X = \{A \cap X \mid A \in \mathfrak{F}\}$  is an open filter base on  $X$  since  $X$  is dense in  $X^*$ . If we consider  $\mathfrak{F} \cap X$  to be a filter base on  $\tilde{X}$ , it has an accumulation point  $F \in \tilde{X}$ . Suppose  $F \cap f^{-1}(W_y) = \emptyset$  for some  $F$  in  $\mathbf{F}$  and neighborhood  $W_y$  of  $y$ . Choose  $A$  in  $\mathfrak{F}$  such that  $f^*(A) \subset W_y$ . Now  $F \cap (A \cap X) \neq \emptyset$  by Lemma 3 of reference [3]; but for  $x \in F \cap (A \cap X)$ , the neighborhood filter  $N_x$  is a member of  $A$  and  $f^*(N_x) = f(x) \in W_y$ , so  $f^{-1}(W_y) \cap F \neq \emptyset$ , a contradiction. Thus  $F \cap f^{-1}(W_y) \neq \emptyset$  for every  $F$  in  $\mathbf{F}$  and every neighborhood  $W_y$  of  $y$ .

We show that  $F_y$  is an accumulation point of  $\mathfrak{F}$ . If not, there is an open  $U \subset X$  such that  $U \in F_y$  and  $U^* \cap A_1 = \emptyset$  for some  $A_1$  in  $\mathfrak{F}$ . Choose  $F$  in  $\mathbf{F}$  and a neighborhood  $W_y$  of  $y$  so that  $F \cap f^{-1}(W_y) \subset U$ . Let  $A_2$  in  $\mathfrak{F}$  be such that  $f^*(A_2) \subset W_y$ , and define  $A = A_1 \cap A_2$ . Then for any  $x \in A \cap X$ , it is true that  $f(x) \in W_y$ . Or in other words,  $A \cap X \subset f^{-1}(W_y)$ , which contradicts our statement that  $U^* \cap A_1$  is empty.

It is clear from (2.1) and (2.2) that  $f^*$  is a compact mapping, so  $(X^*, f^*)$  is a compactification of  $f$ . The next proposition completes the proof of Theorem 3.1.

PROPOSITION 3.6. *For each  $y \in Y$ , there is a homeomorphism  $h_y$  of  $f^{*-1}(y)$  onto  $\tilde{C}(y)$  which leaves the points of  $f^{-1}(y)$  fixed.*

PROOF. Let  $h: X^* \rightarrow \tilde{X}$  be defined by  $h(F_y) = F$ . To show that  $h$  is continuous, let  $U$  be an open set in  $X$  such that  $U \in \mathbf{F} = h(F_y)$ . Choose an open set  $V \in \mathbf{F}$  so that  $\text{cl}(\tilde{V}) \subset \tilde{U}$ . We know that  $F \subset F_y$ , so  $V^*$  is a neighborhood of  $F_y$ . Suppose  $G_z \in V^*$ ; then  $V \in G_z$ , and  $V \cap G \neq \emptyset$  for every  $G \in \mathbf{G}$ , which means that  $G \in \text{cl}(\tilde{V}) \subset \tilde{U}$ . Thus  $h$  is continuous.

For each  $y \in Y$ , let  $h_y = h|_{f^{*-1}(y)}$ . It follows from (3.2) that  $h_y$  is one to one, and thus a homeomorphism since  $f^{*-1}(y)$  is compact. That the

range of  $h_y$  is  $\tilde{C}(y)$  is a consequence of the fact that  $F_y$  is defined if and only if every member of  $F$  meets the inverse image of every neighborhood of  $y$ .

Finally, we show that any compactification  $(X', f')$  of  $f$  with  $X'$  completely regular is a filter space compactification corresponding to some compactification  $\tilde{X}$  of the space  $X$ . We have

**THEOREM 3.7.** *Suppose  $(X', f')$  is a compactification of  $f$  with  $X'$  completely regular, and let  $\tilde{X}$  be a compactification of  $X'$ , and hence of  $X$ . Then the corresponding filter space compactification  $(X^*, f^*)$  is equivalent to  $(X', f')$ .*

**PROOF.** We have  $X \subset X' \subset \tilde{X}$ , and will consider  $\tilde{X}$  (and hence  $X'$ ) to be a filter space over  $X$ . Note that the filter  $F$  on  $X'$  converges to the point  $F \in X'$ , so that for any  $F \in X'$ , we have  $F_y \in X^*$ , where  $y = f'(F)$ .

Let  $F_y$  be an element of  $X^*$ . If we consider  $F_y$  to be a filter on  $X'$ , it has an accumulation point  $G \in X'$ , since  $f(F_y) = f'(F_y)$  converges to  $y$ . Any open set in  $X$  which is a member of  $G$  must meet every member of  $F$ , so it follows that  $F = G$  since  $\tilde{X}$  is Hausdorff. Thus  $f'(F) = y$ . It now follows that  $f(F)$  converges to  $y$ , so that  $F_y = F$ . Thus  $X^* = X'$  and  $f^* = f'$ .

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