COMPACTIFICATION OF MAPPINGS

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1. **Introduction.** A mapping (continuous function) f: $X \to f(X) = Y$ is *compact* if it is closed and point inverses are compact. (These mappings are often called *perfect*, or *proper*.) A compact mapping has the property that the inverse image of any compact set is compact. This property is sometimes taken as the definition of a compact mapping, although the two definitions are not equivalent for arbitrary range Y. Exact conditions for their equivalence are given in reference [**6**].

A *compactification* of f is a pair (X^*, f^*) where X^* is a Hausdorff space containing X as a dense subspace, and f^* is a compact mapping of X^* onto Y such that $f^*|X = f$. G. T. Whyburn introduced the notion of a mapping compactification in 1953 when he showed that every mapping of one locally compact Hausdorff space onto another is a partial mapping of a compact mapping on a Hausdorff space [**4**], [**7**]. He also noted in reference [**7**] that a compactification for any mapping from one completely regular space onto another can be obtained by restricting the extension $\bar{f}$: $\beta X \to \beta Y$ to $\bar{f}^{-1}(Y)$.

In this paper we study some general properties of mapping compactifications and construct a class of so-called filter space compactifications of a mapping from a completely regular space onto a regular space. Each of these compactifications is associated in a natural way with a compactification of the domain of f. For Y locally compact, we show that the domain X is completely regular if and only if f has a compactification; and if X is completely regular, every compactification of f is a filter space compactification. We shall assume that all spaces are Hausdorff. Thus in this paper, regular spaces, completely regular spaces, etc., are Hausdorff.

2. **Mapping compactifications.** Let R denote the collection of all points p in Y having the property that there is a filter $\mathbf{F}$ on X with no accumulation point such that $f(\mathbf{F})$ converges to p. The following result then follows directly from Theorem 1, p. 101, of [**1**].

PROPOSITION 2.1. *The restriction of f to $f^{-1}(Y-R)$ is compact, and if f is compact, R is empty.*

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In case X is regular, we need only consider *open filters*; that is, those which have a base consisting of open sets. Specifically, we have

PROPOSITION 2.2. *If X is regular, R consists of those points p in Y having the property that there is an open filter $\mathbf{G}$ on X with no accumulation point which is such that $f(\mathbf{G})$ converges to p.*

PROOF. Suppose p is in R, $\boldsymbol{F}$ has no accumulation point, and $f(\boldsymbol{F})$ converges to p. Let $\mathcal{V}$ be the collection of all closed neighborhoods in X which have a member of $\boldsymbol{F}$ contained in their complement. The fact that $\boldsymbol{F}$ has no accumulation point guarantees that each x in X has at least one neighborhood in $\mathcal{V}$.

The collection $\mathcal{A} = \{X - V_\alpha \mid V_\alpha \in \mathcal{V}\}$ is a filter base. If $\boldsymbol{A}$ denotes the filter generated by $\mathcal{A}$, then clearly $\boldsymbol{G} = \{A \cap f^{-1}(U) \mid A \in \boldsymbol{A},\ U$ is a neighborhood of $p\}$ is an open filter having no accumulation point, and $f(\boldsymbol{G})$ converges to p.

We shall let S denote the singular set of f. (S is the collection of all p in Y such that in every neighborhood of p there is a compact set with a noncompact inverse image [**2**], [**5**].) It follows easily from the fact that f restricted to $f^{-1}(Y - \mathrm{cl}(R))$ is compact and the results in reference [**6**] that $S \subset \mathrm{cl}(R)$, and that if Y is a k-space then $R \subset S = \mathrm{cl}(R)$.

THEOREM 2.3. *If (X^*, f^*) is a compactification of f, then $f^*(X^* - X) = R$.*

PROOF. Let p be in R and suppose $\boldsymbol{F}$ is a filter on X such that $f(\boldsymbol{F})$ converges to p and $\boldsymbol{F}$ has no accumulation point in X. f^* is compact, so there is an accumulation point x of $\boldsymbol{F}$ in $X^* - X$, and from the continuity of f it follows that $f^*(x) = p$.

For x in $X^* - X$, let $\boldsymbol{N}$ be the neighborhood filter of x. Since X is dense in X^*, $\boldsymbol{F} = \{N \cap X \mid N \in \boldsymbol{N}\}$ is a filter on X without an accumulation point in X and is such that $f(\boldsymbol{F})$ converges to $f^*(x)$.

COROLLARY. *If X is locally compact, R is closed. If in addition, Y is a k-space, then $R = S$.*

PROOF. If X is locally compact, X is open in X^*. Thus $f^*(X^* - X)$ is closed.

A relation $\geqq$ is defined on the collection of all compactifications of f by agreeing that $(X_2, f_2) \geqq (X_1, f_1)$ if there is a mapping h of X_2 onto X_1 which leaves the points of X fixed and is such that $f_2 = f_1 h$. If the mapping h can be taken to be a homeomorphism, (X_1, f_1) and (X_2, f_2) are said to be *equivalent*. It is easy to verify that the relation $\geqq$ is a partial order and that (X_1, f_1) and (X_2, f_2) are equivalent if and only if the relations $(X_2, f_2) \geqq (X_1, f_1) \geqq (X_2, f_2)$ hold.

In case X and Y are locally compact, Whyburn's unified space of the mapping f provides a compactification (X_w, f_w) in which $f_w | (X_w - X)$ is a homeomorphism onto R [4]. The next result shows that any compactification with this property precedes all others with respect to the partial order $\geqq$. Thus all such compactifications of f are equivalent.

THEOREM 2.4. *Suppose the domain of f is locally compact. If (X_w, f_w) is a compactification of f such that f_w restricted to $X_w - X$ is a homeomorphism, then $(X^*, f^*) \geqq (X_w, f_w)$ for any compactification (X^*, f^*) of f.*

PROOF. Define the function $h: X^* \to X_w$ by setting $h(x) = x$ for x in X and $h(x) = [f_w^{-1} f^*(x)] \cap (X_w - X)$ for x in $X^* - X$. It is clear that h is a function, that $f^* = f_w h$, and that h is continuous at all points of X. To show that h is continuous at a point x in $X^* - X$, let $\boldsymbol{N}$ be the neighborhood filter of x. Then $f^*(\boldsymbol{N})$ converges to $f^*(x)$, and f_w is compact, so $h(\boldsymbol{N})$ has at least one accumulation point z. From the fact that X is open in X_w and X^*, and h is the identity on X, it follows that z can not be in X, so $h(\boldsymbol{N})$ has exactly one accumulation point $z = h(x)$ in $X_w - X$.

Let W be a neighborhood of z. If every member of $h(\boldsymbol{N})$ meets the complement of W, then $\boldsymbol{G} = \{ G \cap (X_w - W) | G \in h(\boldsymbol{N}) \}$ is a filter on X_w with no accumulation point, and $f_w(\boldsymbol{G})$ converges to y. This contradicts the fact that f_w is compact. Thus $h(\boldsymbol{N})$ converges to $z = h(x)$ and h is continuous.

3. Filter space compactifications.

Consider a mapping $f: X \to f(X) = Y$, where X is completely regular and Y is regular. Let $\tilde{X}$ be a compactification (Hausdorff) of the space X. For $y \in Y$, let $\tilde{C}(y) \subset \tilde{X}$ be the collection of all accumulation points in $\tilde{X}$ of the inverse image of the neighborhood filter of y. Note that $f^{-1}(y) \subset \tilde{C}(y)$ and $f^{-1}(y) = \tilde{C}(y)$ if and only if $y \in Y - R$.

With the above assumptions and terminology, we now have the following theorem.

THEOREM 3.1. *The mapping f has a compactification (X^*, f^*) with the property that for each $y \in Y$, there is a homeomorphism h_y of $f^{*-1}(y)$ onto $\tilde{C}(y)$ which leaves the points of $f^{-1}(y)$ fixed.*

We shall construct (X^*, f^*) and establish via a series of propositions that it is a compactification of f having the property in the theorem.

In constructing (X^*, f^*), extensive use of the idea of a *filter space* as developed by F. J. Wagner [3] will be made. If $\tilde{X}$ is a collection of open filters on X which includes the neighborhood filter of every x

in X, then $\tilde{X}$ can be made into a topological space by taking the collection of sets $\tilde{U}=\{\boldsymbol{F}\in\tilde{X} \mid U\in\boldsymbol{F},\ U$ is an open subset of $X\}$ as a base for open sets in $\tilde{X}$. The space $\tilde{X}$ is known as a filter space over X. It is shown in reference [3] that the function $e: X\rightarrow e(X)\subset\tilde{X}$ defined by $e(x)=$ the neighborhood filter of x, is a homeomorphism and $e(X)$ is a dense subspace of $\tilde{X}$.

If $\tilde{X}$ is a compactification of the space X, we may consider $\tilde{X}$ to be a filter space over X by associating with each point in $\tilde{X}$ the trace on X of its neighborhood filter (Theorem 2 of [3]). For each filter $\boldsymbol{F}$ in $\tilde{X}$, consider $\mathfrak{A}_y=\{F\cap f^{-1}(W_y) \mid F\in\boldsymbol{F},\ W_y$ a neighborhood of $y\}$. Each $\mathfrak{A}_y$ that does not contain the empty set is a base for a filter on X, which we shall denote by $\boldsymbol{F}_y$. Note that $\boldsymbol{F}_y$ is defined if and only if $F\cap f^{-1}(W_y)\neq\varnothing$ for every neighborhood W_y of y and every $F\in\boldsymbol{F}$.

To avoid confusion, we shall throughout use the terminology "$\boldsymbol{F}$ is a filter *on* X" when $\boldsymbol{F}$ is a collection of subsets of X, and "$\boldsymbol{F}$ is a filter *in* X" when $\boldsymbol{F}$ is a member of X.

PROPOSITION 3.2. *For* $\boldsymbol{F}$ *and* $\boldsymbol{G}$ *in* $\tilde{X}$, $\boldsymbol{F}_y=\boldsymbol{G}_z$ *if and only if* $\boldsymbol{F}=\boldsymbol{G}$, *and* $y=z$.

PROOF. The "if" part is trivial. Assume $\boldsymbol{F}_y=\boldsymbol{G}_z$. The inverse image of every neighborhood of y is a member of $\boldsymbol{F}_y$ and the inverse image of every neighborhood of z is a member of $\boldsymbol{G}_z$, so it follows from the fact that Y is Hausdorff that $y=z$. That $\boldsymbol{F}=\boldsymbol{G}$ follows from the fact that $\boldsymbol{F}\subset\boldsymbol{F}_y$, $\boldsymbol{G}\subset\boldsymbol{G}_z$, and the Hausdorff property of $\tilde{X}$.

It is clear that each $\boldsymbol{F}_y$ is an open filter and that for $\boldsymbol{F}=\boldsymbol{N}_x$, the neighborhood filter of x, then $\boldsymbol{F}_y=\boldsymbol{N}_x$, where $y=f(x)$. Let X^* denote the filter space consisting of all such $\boldsymbol{F}_y$. Define the function f^* from X^* onto Y by setting $f^*(\boldsymbol{F}_y)=y$. Then $f^*|X=f$ and X is dense in X^*.

PROPOSITION 3.3. f^* *is continuous.*

PROOF. Let R be a neighborhood of $y=f^*(\boldsymbol{F}_y)$ and choose W to be an open neighborhood of y such that $\mathrm{cl}(W)\subset R$. Now $U=f^{-1}(W)$ is a member of $\boldsymbol{F}_y$, so $U^*=\{\boldsymbol{G}_z\in X^* \mid U\in\boldsymbol{G}_z\}$ is a neighborhood of $\boldsymbol{F}_y$. For any $\boldsymbol{G}_z\in U^*$, $f^{-1}(W)\in\boldsymbol{G}_z$, so $z=f^*(\boldsymbol{G}_z)\in\mathrm{cl}(W)\subset R$.

PROPOSITION 3.4. X^* *is regular.*

PROOF. Let $\boldsymbol{F}_y$ and $\boldsymbol{G}_z$ be two distinct points in X^*. If $y\neq z$, the existence of disjoint neighborhoods of the two points follows from the continuity of f^* and the Hausdorff property of Y. If $\boldsymbol{F}=\boldsymbol{G}$, there are disjoint open sets U and V in X such that $U\in\boldsymbol{F}$ and $V\in\boldsymbol{G}$ since $\tilde{X}$ is Hausdorff. Thus U^* and V^* are disjoint neighborhoods of $\boldsymbol{F}_y$ and $\boldsymbol{G}_z$, respectively.

Next, let $\boldsymbol{F}_y \in U^*$, and choose an open $F \in \boldsymbol{F}$ and a neighborhood W_y of y so that $F \cap f^{-1}(W_y) \subset U$. Let M be an open neighborhood of y such that $\text{cl}(M) \subset W_y$. The space $\tilde{X}$ is regular, so choose an open $V \subset X$ so that $V \in \boldsymbol{F}$ and $\text{cl}(\tilde{V}) \subset \tilde{F}$, where $\tilde{V}$ denotes the collection of all elements in $\tilde{X}$ that have V as a member. Let $R = V \cap f^{-1}(M)$. We shall show that R^* is a neighborhood of $\boldsymbol{F}_y$ such that $\text{cl}(R^*) \subset U^*$.

First note that since $V \in \boldsymbol{F}$, we have $V \cap f^{-1}(M) = R \in \boldsymbol{F}_y$, so R^* is indeed a neighborhood of $\boldsymbol{F}_y$. If $\boldsymbol{G}_z \in \text{cl}(R^*)$, every open set in $\boldsymbol{G}_z$ meets R, so $z \in \text{cl}(M) \subset W_y$. Let W_z be a neighborhood of z such that $W_z \subset W_y$. For every $G \in \boldsymbol{G}$, we have $[G \cap f^{-1}(W_z)] \cap R \neq \varnothing$, and hence $G \cap V \neq \varnothing$ for every G in $\boldsymbol{G}$. This means that $\boldsymbol{G} \in \text{cl}(\tilde{V}) \subset \tilde{F}$, or $F \in \boldsymbol{G}$. Hence $F \cap f^{-1}(W_z) \in \boldsymbol{G}_z$, and $U \in \boldsymbol{G}_z$ since $F \cap f^{-1}(W_z) \subset U$.

Proposition 3.5. *If $\mathfrak{F}$ is an open filter on X^* such that $f^*(\mathfrak{F})$ converges to some y in Y, then $\mathfrak{F}$ has an accumulation point.*

Proof. $\mathfrak{F} \cap X = \{A \cap X \mid A \in \mathfrak{F}\}$ is an open filter base on X since X is dense in X^*. If we consider $\mathfrak{F} \cap X$ to be a filter base on $\tilde{X}$, it has an accumulation point $\boldsymbol{F} \in \tilde{X}$. Suppose $F \cap f^{-1}(W_y) = \varnothing$ for some F in $\boldsymbol{F}$ and neighborhood W_y of y. Choose A in $\mathfrak{F}$ such that $f^*(A) \subset W_y$. Now $F \cap (A \cap X) \neq \varnothing$ by Lemma 3 of reference [3]; but for $x \in F \cap (A \cap X)$, the neighborhood filter $\boldsymbol{N}_x$ is a member of A and $f^*(\boldsymbol{N}_x) = f(x) \in W_y$, so $f^{-1}(W_y) \cap F \neq \varnothing$, a contradiction. Thus $F \cap f^{-1}(W_y) \neq \varnothing$ for every F in $\boldsymbol{F}$ and every neighborhood W_y of y.

We show that $\boldsymbol{F}_y$ is an accumulation point of $\mathfrak{F}$. If not, there is an open $U \subset X$ such that $U \in \boldsymbol{F}_y$ and $U^* \cap A_1 = \varnothing$ for some A_1 in $\mathfrak{F}$. Choose F in $\boldsymbol{F}$ and a neighborhood W_y of y so that $F \cap f^{-1}(W_y) \subset U$. Let A_2 in $\mathfrak{F}$ be such that $f^*(A_2) \subset W_y$, and define $A = A_1 \cap A_2$. Then for any $x \in A \cap X$, it is true that $f(x) \in W_y$. Or in other words, $A \cap X \subset f^{-1}(W_y)$, which contradicts our statement that $U^* \cap A_1$ is empty.

It is clear from (2.1) and (2.2) that f^* is a compact mapping, so (X^*, f^*) is a compactification of f. The next proposition completes the proof of Theorem 3.1.

Proposition 3.6. *For each $y \in Y$, there is a homeomorphism h_y of $f^{*-1}(y)$ onto $\tilde{C}(y)$ which leaves the points of $f^{-1}(y)$ fixed.*

Proof. Let $h: X^* \to \tilde{X}$ be defined by $h(\boldsymbol{F}_y) = \boldsymbol{F}$. To show that h is continuous, let U be an open set in X such that $U \in \boldsymbol{F} = h(\boldsymbol{F}_y)$. Choose an open set $V \in \boldsymbol{F}$ so that $\text{cl}(\tilde{V}) \subset \tilde{U}$. We know that $\boldsymbol{F} \subset \boldsymbol{F}_y$, so V^* is a neighborhood of $\boldsymbol{F}_y$. Suppose $\boldsymbol{G}_z \in V^*$; then $V \in \boldsymbol{G}_z$, and $V \cap G \neq \varnothing$ for every $G \in \boldsymbol{G}$, which means that $\boldsymbol{G} \in \text{cl}(\tilde{V}) \subset \tilde{U}$. Thus h is continuous.

For each $y \in Y$, let $h_y = h \mid f^{*-1}(y)$. It follows from (3.2) that h_y is one to one, and thus a homeomorphism since $f^{*-1}(y)$ is compact. That the range of h_y is $\tilde{C}(y)$ is a consequence of the fact that $\boldsymbol{F}_y$ is defined if and only if every member of $\boldsymbol{F}$ meets the inverse image of every neighborhood of y.

Finally, we show that any compactification (X', f') of f with X' completely regular is a filter space compactification corresponding to some compactification $\tilde{X}$ of the space X. We have

THEOREM 3.7. *Suppose (X', f') is a compactification of f with X' completely regular, and let $\tilde{X}$ be a compactification of X', and hence of X. Then the corresponding filter space compactification (X^*, f^*) is equivalent to (X', f').*

PROOF. We have $X \subset X' \subset \tilde{X}$, and will consider $\tilde{X}$ (and hence X') to be a filter space over X. Note that the filter $\boldsymbol{F}$ on X' converges to the point $\boldsymbol{F} \in X'$, so that for any $\boldsymbol{F} \in X'$, we have $\boldsymbol{F}_y \in X^*$, where $y = f'(\boldsymbol{F})$.

Let $\boldsymbol{F}_y$ be an element of X^*. If we consider $\boldsymbol{F}_y$ to be a filter on X', it has an accumulation point $\boldsymbol{G} \in X'$, since $f(\boldsymbol{F}_y) = f'(\boldsymbol{F}_y)$ converges to y. Any open set in X which is a member of $\boldsymbol{G}$ must meet every member of $\boldsymbol{F}$, so it follows that $\boldsymbol{F} = \boldsymbol{G}$ since $\tilde{X}$ is Hausdorff. Thus $f'(\boldsymbol{F}) = y$. It now follows that $f(\boldsymbol{F})$ converges to y, so that $\boldsymbol{F}_y = \boldsymbol{F}$. Thus $X^* = X'$ and $f^* = f'$.

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