POINT-TRANSITIVE ACTIONS BY A
STANDARD METRIC THREAD

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1. Introduction. An action is a continuous function \( \alpha: T \times X \rightarrow X \) where \( T \) is a (topological) semigroup, \( X \) is a Hausdorff space, and \( \alpha(t_1t_2, x) = \alpha(t_1, \alpha(t_2, x)) \) for all \( t_1, t_2 \in T \) and \( x \in X \). If \( Q(\alpha) = \{ x \mid \alpha(T \times \{ x \}) = X \} \) is not empty, then \( \alpha \) is called a point-transitive action. Actions by semigroups have been studied in [1], [2], [4] and [9]. The purpose of this note is to describe how the point-transitive actions of a standard metric thread may be constructed from point-transitive actions of usual, nil, and \( I \)-semigroups which have been classified in [2]. The reader is referred to [6], [8], and [10] for general information about the theory of semigroups.

A standard metric thread is a semigroup which is homeomorphic to a closed interval of real numbers and which has an identity at one endpoint and a zero at the other. This type of semigroup has been studied in [5] and [7]. A usual \( I \)-semigroup is a semigroup isomorphic (topologically isomorphic) to \([0, 1]\) under the usual multiplication; a nil \( I \)-semigroup is a semigroup isomorphic to \([0, 1]/[0, \frac{1}{2}]\); a min \( I \)-semigroup is a semigroup isomorphic to \([0, 1]\) under the multiplication \( xy = \min(x, y) \). We shall have occasion to use the following theorem due to Mostert and Shields [7].

Theorem 1 [7]. Let \( T \) be a standard metric thread and \( E \) be the set of idempotents. If \( x, y \in E \) then \( xy = \min(x, y) \). Let \( C \) be the closure of a component of the complement of \( E \) in \( T \). Then \( C \) is a usual \( I \)-semigroup or a nil \( I \)-semigroup. If \( x \in C \) and \( y \in C \) then \( xy = \min(x, y) \).

A function \( f \) from a semigroup to a Hausdorff space is said to be a multiplicative function if and only if \( f(t_1t_2) = f(t_2) \) implies \( f(t_1) = f(t_2) \) and \( f(t_1t) = f(t_2t) \) for all \( t \) in the semigroup. The following theorem which is proved in [1] is a valuable aid in the study of point-transitive actions by Abelian semigroups.

Theorem 2 [1]. Let \( T \) be a compact, Abelian semigroup and let \( X \) be a compact, Hausdorff space.

1. If \( h: T \rightarrow X \) is a multiplicative function, then \( tx = h(\theta^{-1}(x)) \) defines an action of \( T \) on \( X \).

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(2) If \( T \times X \to X \) is an action and \( x \in Q = \{ y \in X \midTy = X \} \), then the function \( h: T \to X \) defined by \( h(t) = tx \) is multiplicative. Moreover, \( ty = h(th^{-1}(y)) \) for all \( t \in T \) and \( y \in X \).

2. Construction. We shall use the following notation through the remainder of this note. Let \( T \) be a standard metric thread with zero 0 and identity \( e; E = \{ x \in T \mid x^2 = x \} \), i.e., the set of all idempotents of \( T \); \( \mathcal{E} = \{ C \subseteq T \mid C \text{ is the closure of a component of } T \setminus E \text{ or } C \text{ is a nondegenerate interval contained in } E \} \); \( F = \{ \inf C \mid C \in \mathcal{E} \} \); for each \( x \in F \) let \( C_x \subseteq \mathcal{E} \) such that \( x = \inf C_x \); if \( x = \inf C_x \) then \( x + 1 = \sup C_x \); and let \( Z = F \cup (T \setminus \cup C) \cup \{ e \} \). We shall assume an ordering \( \leq \) on \( T \) defined by \( t_1 \leq t_2 \) if and only if \( t_1 = 0 \) or \( t_1 \) separates 0 from \( t_2 \).

**Theorem 3.** Let \( g: Z \to [0, 1] \) be a continuous order preserving function with \( g(z) = 0 \) and \( g(e) = 1 \). For each \( x \in F \) let \( \alpha_x \) be a point-transitive action of \( C_x \) on \( [g(x), g(x + 1)] \) with \( g(x + 1) \in Q(\alpha_x) \) = \( \{ z \in [g(x), g(x + 1)] \mid \alpha_x(C_x \times \{ z \}) = [g(x), g(x + 1)] \} \). Define \( \alpha: T \times [0, 1] \to [0, 1] \) by

\[
\alpha(t, z) = \begin{cases} 
\alpha_x(t, z) & \text{if } t \in C_x \text{ and } g(x) \leq z \leq g(x + 1), \\
\alpha_x(t, x + 1) & \text{if } t \in C_x \text{ and } g(x + 1) < z, \\
z & \text{if } t \in C_x \text{ and } z < g(x), \\
g(t) & \text{if } t \in T \setminus \cup C \text{ and } g(t) \leq z, \\
z & \text{if } t \in T \setminus \cup C \text{ and } z < g(t). 
\end{cases}
\]

Then \( \alpha \) is a point-transitive action with \( 1 \in Q(\alpha) \).

**Proof.** For \( x \in F \) let \( h_x: C_x \to [g(x), g(x + 1)] \) be defined by \( h_x(t) = \alpha_x(t, g(x + 1)) \). Define \( h: T \to [0, 1] \) by

\[
h(t) = \begin{cases} 
h_x(t) & \text{if } t \in C_x, \\
g(t) & \text{if } t \in (T \setminus \cup C). 
\end{cases}
\]

We shall show that \( h \) is a continuous multiplicative function and \( \alpha(t, z) = h(th^{-1}(z)) \). The proof is divided into the following five lemmas.

**Lemma 1.** (1) If \( x \in F \), then \( h_x(x) = g(x) \) and \( h_x(x + 1) = g(x + 1) \).

(2) Let \( t_1 \in Z \), \( t_2 \in T \). If \( t_1 \leq t_2 \) (\( t_2 \leq t_1 \)), then \( g(t_1) \leq h(t_2) \) (\( h(t_2) \leq g(t_1) \)).

**Proof.** (1) follows from the results in [3].

(2) If \( t_0 \in Z \), then (2) follows from the fact that \( g \) is order preserving. If \( t \in C_x \) then \( h(t) = h_x(t) \geq h_x(x) = g(x) \geq g(t_1) \) since \( t_1 \leq x \).

**Lemma 2.** The function \( h \) is continuous.
Proof. Suppose \( \{ t_n \} \) is a sequence which converges to \( t \). It will be shown that \( \{ h(t_n) \} \) converges to \( h(t) \). If there is an \( x \in F \) such that \( t \in \text{Interior } C_x \), then it is clear that \( \{ h(t_n) \} \) converges to \( h(t) \).

If \( t \in F \), then \( \{ t_n \} \) is residually in \( C_t \) or \( C_{t-1} \), in which case \( \{ h(t_n) \} \) converges to \( h(t) \) or \( \{ t_n \} \) is cofinally in \( C_t \) or \( C_{t-1} \). In the latter case, if \( q \) is a cluster point of \( \{ h(t_n) \} \), then \( q = h(t) \) since \( h_{t-1} \) and \( h_t \) are continuous.

Suppose \( t \in (T \setminus U \cap \mathcal{E}) \). Let \( q \) be a cluster point of \( h(t_n) \) and \( h(t'_n) \) be a subsequence of \( h(t_n) \) which converges to \( q \). If \( t_n' \) is cofinally in \( Z \), then there is a subsequence \( t_n'' \) of \( t_n' \) which is in \( Z \). Since \( \{ t_n'' \} \) converges to \( t \), \( \{ g(t_n'') \} \) converges to \( g(t) = h(t) \). Thus, we may suppose that \( \{ t_n' \} \) is in \( T \setminus Z \) so that for each \( n \) there is \( x_n \in F \) such that \( t_n' \in C_{x_n} \).

It follows that \( \{ x_n \} \) and \( \{ x_n + 1 \} \) converge to \( t \). Thus, \( h(t_n') \) converges to \( h(t) = g(t) \) since \( g(x_n) \leq h(t_n') \leq g(x_n + 1) \).

The remaining case, \( t = e \in \mathcal{U} \cap \mathcal{E} \), is clear.

**Lemma 3.** The function \( h \) is monotone.

Proof. To show \( h \) is monotone it suffices to show that \( h(t_1) = h(t_2) \) and \( t_1 < t < t_2 \) implies \( h(t) = h(t_1) = h(t_2) \).

Suppose \( t_1 \in C_{x_1} \) and \( t_2 \in C_{x_2} \). Since \( h(t_1) \in [g(x_1), g(x_1 + 1)] \) and \( h(t_2) \in [g(x_2), g(x_2 + 1)] \), \( h_{x_1} \) and \( h_{x_2} \) are monotone [3], and \( g \) is order preserving; \( h(t_1) \leq g(x_1 + 1) \leq g(t_2) \leq h(t_2) \). If \( t \in C_{x_1} \), \( h(t_1) \leq h(t) \leq g(x_1 + 1) \). If \( x_1 + 1 \leq t \leq x_2 \), then \( g(x_1 + 1) \leq h(t) \leq g(x_2) \). If \( t \in C_{x_2} \), then \( g(x_2) \leq h(t) \leq h(t_2) \). Thus in this case \( h(t) = h(t_1) = h(t_2) \).

Suppose \( t_1 \in (T \setminus \mathcal{U} \cap \mathcal{E}) \) and \( t_2 \in C_{x_2} \). Since \( t_1 \leq x_2 \), \( g(t_1) \leq g(x_2) \). Thus, \( g(x_2) \leq h(t) \leq h(t_2) \) for \( x_2 \leq t \leq t_2 \) and \( g(t_1) \leq h(t) \leq g(x_2) \) for \( t_1 \leq t \leq x_2 \).

The other cases are similar.

**Lemma 4.** The function \( h \) is multiplicative.

Proof. Suppose \( h(t_1) = h(t_2) \), \( t_1 < t_2 \) and \( t \in T \). The proof that \( h(tt_1) = h(tt_2) \) is divided into three cases.

1. \( t_1 \leq t \leq t_2 \). If \( t_1 \in E \), then \( t_1 = tt_1 \leq tt_2 \leq t_2 \) so that \( h(tt_1) = h(tt_2) \). If \( t_1 \in E \) then \( t_1 \in C_{x_1} \) for some \( x_1 \in F \). Since \( h(x_1 + 1) = h(t_1) \) and because of the results in [2], \( h(x_1) = h(t_1) \). Thus, \( h(t_1) = h(x_1) = h(tt_1) \leq h(tt_2) \).

2. \( t_2 < t \). If \( t_1, t_2 \in C_x \) for some \( x \in F \), then \( h(tt_1) = h(tt_2) \) since \( h \big| C_x \) is multiplicative. If \( t_1 \) and \( t_2 \) are not both in \( C_x \) for all \( x \in F \), then \( t_1 = tt_1 \leq tt_2 \leq t_2 \) and \( h(tt_1) = h(tt_2) \) by Lemma 3.

3. \( t < t_1 \) is similar to (2).

The following lemma concludes the proof of Theorem 3.
**Lemma 5.** \[ \alpha(t, z) = h(th^{-1}(z)) \text{ and } 1 \in Q(\alpha). \]

**Proof.** The first statement follows from straightforward applications of Theorem 1 and the definitions of \( \alpha \) and \( h \).

The second statement follows from the following equalities:
\[ 1 = h(e), [0, 1] = h(T) = h(Te). \]

Next we shall show that the method of construction described in Theorem 3 gives all of the point-transitive actions by \( T \). It is shown in [2] that if \( \alpha: T \times X \rightarrow X \) is a point-transitive action, then \( X \) is homeomorphic to \([0, 1]\), and \( Q(\alpha) = 1 \) and \( Q(\alpha) \) is contained in the set of endpoints of \( X \). Thus, without loss of generality we may assume that if \( \alpha: T \times X \rightarrow X \) is a point-transitive action, then \( X = [0, 1] \) and \( Q(\alpha) = \{1\} \).

**Proposition 4.** Let \( \alpha: T \times [0, 1] \rightarrow [0, 1] \) be a point-transitive action with \( 1 \in Q \). Then there is a continuous order preserving function \( g: Z \rightarrow [0, 1] \) such that \( g(z) = 0 \) and \( g(e) = 1 \) and such that
\[ \alpha(C_x \times [g(x), g(x + 1)]) = [g(x), g(x + 1)] \]
for \( x \in F \). Moreover, for \( x \in F \), \( \alpha_x = \alpha| (C_x \times [g(x), g(x + 1)]) \) is a point-transitive action of \( C_x \) on \([g(x), g(x + 1)]\) with \( g(x + 1) \in Q(\alpha_x) \) and
\[
\begin{align*}
\alpha(t, z) &= \alpha_x(t, z) & \text{if } t \in C_x & \text{and } g(x) \leq z \leq g(x + 1), \\
&= \alpha_x(t, x + 1) & \text{if } t \in C_x & \text{and } g(x + 1) < z, \\
&= z & \text{if } t \in C_x & \text{and } z < g(x), \\
&= g(t) & \text{if } t \in T \setminus U & \text{and } g(t) \leq z, \\
&= z & \text{if } t \in T \setminus U & \text{and } z < g(t).
\end{align*}
\]

**Proof.** Let \( h: T \rightarrow [0, 1] \) be defined by \( h(t) = \alpha(t, 1) \), and let \( g = h| z \). Since \( h(e) = 1 \), \( h(z) = 0 \), and \( h \) is monotone [3], \( g \) is order preserving. Also \( h(C_x \times [g(x), g(x + 1)]) = h(C_x) = [g(x), g(x + 1)] \). So that \( \alpha_x \) is a point-transitive action with \( g(x + 1) \in Q(\alpha_x) \). The remainder of the proof is a straightforward application of Theorem 1.

**Bibliography**


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