POINT-TRANSITIVE ACTIONS BY A STANDARD METRIC THREAD

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1. **Introduction.** An action is a continuous function α: $T\times X\to X$ where T is a (topological) semigroup, X is a Hausdorff space, and $\alpha(t_1t_2, x)=\alpha(t_1, \alpha(t_2, x))$ for all t_1, $t_2\in T$ and $x\in X$. If $Q(\alpha) = \{x \mid \alpha(T\times \{x\})=X\}$ is not empty, then α is called a point-transitive action. Actions by semigroups have been studied in [1], [2], [4] and [9]. The purpose of this note is to describe how the point-transitive actions of a standard metric thread may be constructed from point-transitive actions of usual, nil, and min I-semigroups which have been classified in [2]. The reader is referred to [6], [8], and [10] for general information about the theory of semigroups.

A standard metric thread is a semigroup which is homeomorphic to a closed interval of real numbers and which has an identity at one endpoint and a zero at the other. This type of semigroup has been studied in [5] and [7]. A usual I-semigroup is a semigroup iseomorphic (topologically isomorphic) to $[0, 1]$ under the usual multiplication; a nil I-semigroup is a semigroup iseomorphic to $[0, 1]/[0, \frac{1}{2}]$; a min I-semigroup is a semigroup iseomorphic to $[0, 1]$ under the multiplication $xy=\min(x, y)$. We shall have occasion to use the following theorem due to Mostert and Shields [7].

THEOREM 1 [7]. *Let T be a standard metric thread and E be the set of idempotents. If x, $y\in E$ then $xy=\min(x, y)$. Let C be the closure of a component of the complement of E in T. Then C is a usual I-semigroup or a nil I-semigroup. If $x\in C$ and $y\notin C$ then $xy=\min(x, y)$.*

A function f from a semigroup to a Hausdorff space is said to be a multiplicative function if and only if $f(t_1)=f(t_2)$ implies $f(tt_1)=f(tt_2)$ and $f(t_1t)=f(t_2t)$ for all t in the semigroup. The following theorem which is proved in [1] is a valuable aid in the study of point-transitive actions by Abelian semigroups.

THEOREM 2 [1]. *Let T be a compact, Abelian semigroup and let X be a compact, Hausdorff space.*

(1) *If h: $T\to X$ is a multiplicative function, then $tx=h(th^{-1}(x))$ defines an action of T on X.*

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(2) *If* $T\times X\to X$ *is an action and* $x\in Q=\{y\in X \mid Ty=X\}$, *then the function* h: $T\to X$ *defined by* $h(t)=tx$ *is multiplicative. Moreover,* $ty=h(th^{-1}(y))$ *for all* $t\in T$ *and* $y\in X$.

2. **Construction.** We shall use the following notation through the remainder of this note. Let T be a standard metric thread with zero 0 and identity e; $E=\{x\in T \mid x^2=x\}$, i.e., the set of all idempotents of T; $\mathcal{C}=\{C\subset T \mid C$ is the closure of a component of $T\backslash E$ or C is a nondegenerate interval contained in $E\}$; $F=\{\inf C \mid C\in\mathcal{C}\}$; for each $x\in F$ let $C_x\in\mathcal{C}$ such that $x=\inf C_x$; if $x=\inf C_x$ then $x+1=\sup C_x$; and let $Z=F\cup(T\backslash\bigcup C)\cup\{e\}$. We shall assume an ordering $\leqq$ on T defined by $t_1\leqq t_2$ if and only if $t_1=0$ or t_1 separates 0 from t_2.

THEOREM 3. *Let* g: $Z\to[0, 1]$ *be a continuous order preserving function with* $g(z)=0$ *and* $g(e)=1$. *For each* $x\in F$ *let* α_x *be a point-transitive action of* C_x *on* $[g(x), g(x+1)]$ *with* $g(x+1)\in Q(\alpha_x)$ $=\{z\in[g(x), g(x+1)] \mid \alpha_x(C_x\times\{z\})=[g(x), g(x+1)]\}$. *Define* α: $T\times[0, 1]\to[0, 1]$ *by*

$$\begin{aligned}
\alpha(t, z) &= \alpha_x(t, z) && \textit{if } t\in C_x && \textit{and} && g(x)\leqq z\leqq g(x+1),\\
&= \alpha_x(t, x+1) && \textit{if } t\in C_x && \textit{and} && g(x+1)<z,\\
&= z && \textit{if } t\in C_x && \textit{and} && z<g(x),\\
&= g(t) && \textit{if } t\in T\backslash\bigcup\mathcal{C} && \textit{and} && g(t)\leqq z,\\
&= z && \textit{if } t\in T\backslash\bigcup\mathcal{C} && \textit{and} && z<g(t).
\end{aligned}$$

Then α *is a point-transitive action with* $1\in Q(\alpha)$.

PROOF. For $x\in F$ let h_x: $C_x\to[g(x), g(x+1)]$ be defined by $h_x(t)$ $=\alpha_x(t, g(x+1))$. Define h: $T\to[0, 1]$ by

$$\begin{aligned}
h(t) &= h_x(t) && \text{if } t\in C_x,\\
&= g(t) && \text{if } t\in(T\backslash\bigcup\mathcal{C}).
\end{aligned}$$

We shall show that h is a continuous multiplicative function and $\alpha(t, z)=h(th^{-1}(z))$. The proof is divided into the following five lemmas.

LEMMA 1. (1) *If* $x\in F$, *then* $h_x(x)=g(x)$ *and* $h_x(x+1)=g(x+1)$.
(2) *Let* $t_1\in Z$, $t_2\in T$. *If* $t_1\leqq t_2$ $(t_2\leqq t_1)$, *then* $g(t_1)\leqq h(t_2)$ $(h(t_2)\leqq g(t_1))$.

PROOF. (1) follows from the results in [**3**].
(2) If $t_2\in Z$, then (2) follows from the fact that g is order preserving. If $t\in C_x$ then $h(t)=h_x(t)\geqq h_x(x)=g(x)\geqq g(t_1)$ since $t_1\leqq x$.

LEMMA 2. *The function* h *is continuous.*

PROOF. Suppose $\{t_n\}$ is a sequence which converges to t. It will be shown that $\{h(t_n)\}$ converges to $h(t)$. If there is an $x \in F$ such that $t \in$ Interior C_x, then it is clear that $\{h(t_n)\}$ converges to $h(t)$.

If $t \in F$, then $\{t_n\}$ is residually in C_t or C_{t-1}, in which case $\{h(t_n)\}$ converges to $h(t)$ or $\{t_n\}$ is cofinally in C_t or C_{t-1}. In the latter case, if q is a cluster point of $\{h(t_n)\}$, then $q = h(t)$ since h_{t-1} and h_t are continuous.

Suppose $t \in (T \backslash \bigcup \mathcal{C})$. Let q be a cluster point of $h(t_n)$ and $h(t_n')$ be a subsequence of $h(t_n)$ which converges to q. If t_n' is cofinally in Z, then there is a subsequence t_n'' of t_n' which is in Z. Since $\{t_n''\}$ converges to t, $\{g(t_n'')\}$ converges to $g(t) = h(t)$. Thus, we may suppose that $\{t_n'\}$ is in $T \backslash Z$ so that for each n there is $x_n \in F$ such that $t_n' \in C_{x_n}$. It follows that $\{x_n\}$ and $\{x_n + 1\}$ converge to t. Thus, $h(t_n')$ converges to $h(t) = g(t)$ since $g(x_n) \leqq h(t_n') \leqq g(x_n + 1)$.

The remaining case, $t = e \in \bigcup \mathcal{C}$, is clear.

LEMMA 3. *The function h is monotone.*

PROOF. To show h is monotone it suffices to show that $h(t_1) = h(t_2)$ and $t_1 < t < t_2$ implies $h(t) = h(t_1) = h(t_2)$.

Suppose $t_1 \in C_{x_1}$ and $t_2 \in C_{x_2}$. Since $h(t_1) \in [g(x_1), g(x_1+1)]$ and $h(t_2) \in [g(x_2), g(x_2+1)]$, hx_1 and hx_2 are monotone [**3**], and g is order preserving; $h(t_1) \leqq g(x_1+1) \leqq g(t_2) \leqq h(t_2)$. If $t \in C_{x_1}$, $h(t_1) \leqq h(t) \leqq g(x_1+1)$. If $x_1 + 1 \leqq t \leqq x_2$, then $g(x_1+1) \leqq h(t) \leqq g(x_2)$. If $t \in C_{x_2}$ then $g(x_2) \leqq h(t) \leqq h(t_2)$. Thus in this case $h(t) = h(t_1) = h(t_2)$.

Suppose $t_1 \in (T \backslash \bigcup \mathcal{C})$ and $t_2 \in C_{x_2}$. Since $t_1 \leqq x_2$, $g(t_1) \leqq g(x_2) \leqq h(t_2)$. Thus, $g(x_2) \leqq h(t) \leqq h(t_2)$ for $x_2 \leqq t \leqq t_2$ and $g(t_1) \leqq h(t) \leqq g(x_2)$ for $t_1 \leqq t \leqq x_2$.

The other cases are similar.

LEMMA 4. *The function h is multiplicative.*

PROOF. Suppose $h(t_1) = h(t_2)$, $t_1 < t_2$ and $t \in T$. The proof that $h(tt_1) = h(tt_2)$ is divided into three cases.

(1) $t_1 \leqq t \leqq t_2$. If $t_1 \in E$, then $t_1 = tt_1 \leqq tt_2 \leqq t_2$ so that $h(tt_1) = h(tt_2)$. If $t_1 \notin E$ then $t_1 \in C_{x_1}$ for some $x_1 \in F$. Since $h(x_1+1) = h(t_1)$ and because of the results in [**2**], $h(x_1) = h(t_1)$. Thus, $h(t_1) = h(x_1) = h(tx_1) \leqq h(tt_2) \leqq h(t_2)$.

(2) $t_2 < t$. If $t_1, t_2 \in C_x$ for some $x \in F$, then $h(tt_1) = h(tt_2)$ since $h | C_x$ is multiplicative. If t_1 and t_2 are not both in C_x for all $x \in F$, then $t_1 = tt_1 \leqq tt_2 \leqq t_2$ and $h(tt_1) = h(tt_2)$ by Lemma 3.

(3) $t < t_1$ is similar to (2).

The following lemma concludes the proof of Theorem 3.

LEMMA 5. *$\alpha(t, z) = h(th^{-1}(z))$ and $1 \in Q(\alpha)$.*

PROOF. The first statement follows from straightforward applications of Theorem 1 and the definitions of α and h.

The second statement follows from the following equalities: $1 = h(e)$, $[0, 1] = h(T) = h(Te)$.

Next we shall show that the method of construction described in Theorem 3 gives all of the point-transitive actions by T. It is shown in [2] that if α: $T \times X \to X$ is a point-transitive action, then X is homeomorphic to $[0, 1]$, and $Q(\alpha) = 1$ and $Q(\alpha)$ is contained in the set of endpoints of X. Thus, without loss of generality we may assume that if α: $T \times X \to X$ is a point-transitive action, then $X = [0, 1]$ and $Q(\alpha) = \{1\}$.

PROPOSITION 4. *Let α: $T \times [0, 1] \to [0, 1]$ be a point-transitive action with $1 \in Q$. Then there is a continuous order preserving function g: $Z \to [0, 1]$ such that $g(z) = 0$ and $g(e) = 1$ and such that*

$$\alpha(C_x \times [g(x), g(x + 1)]) = [g(x), g(x + 1)]$$

for $x \in F$. Moreover, for $x \in F$, $\alpha_x = \alpha | (C_x \times [g(x), g(x+1)])$ is a point-transitive action of C_x on $[g(x), g(x+1)]$ with $g(x+1) \in Q(\alpha_x)$ and

$$\begin{aligned}
\alpha(t, z) &= \alpha_x(t, z) && \text{if } t \in C_x && \text{and } g(x) \leqq z \leqq g(x + 1), \\
&= \alpha_x(t, x + 1) && \text{if } t \in C_x && \text{and } g(x + 1) < z, \\
&= z && \text{if } t \in C_x && \text{and } z < g(x), \\
&= g(t) && \text{if } t \in T \backslash \bigcup \mathcal{C} && \text{and } g(t) \leqq z, \\
&= z && \text{if } t \in T \backslash \bigcup \mathcal{C} && \text{and } z < g(t).
\end{aligned}$$

PROOF. Let h: $T \to [0, 1]$ be defined by $h(t) = \alpha(t, 1)$, and let $g = h | z$. Since $h(e) = 1$, $h(z) = 0$, and h is monotone [3], g is order preserving. Also $h(C_x \times [g(x), g(x+1)]) = h(C_x) = [g(x), g(x+1)]$. So that α_x is a point-transitive action with $g(x+1) \in Q(\alpha_x)$. The remainder of the proof is a straightforward application of Theorem 1.

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