UNKNOTTING IN CODIMENSION ONE

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Two PL (=Piecewise linear) embeddings f, g of a polyhedron X_1 into a polyhedron X_2 are *equivalent* if there exists a PL-homeomorphism h of X_2 onto X_2 such that $hf=g$. Let Y_i be a subpolyhedron of X_i, $i=1, 2$; a PL-map f of the pair (X_1, Y_1) into (X_2, Y_2) is a PL-map f: $X_1 \to X_2$ such that $f^{-1}(Y_2)=Y_1$. Two PL-embeddings, f, g, of the pair (X_1, Y_1) into (X_2, Y_2) are *equivalent* if there exists a PL-homeomorphism h of the pair (X_2, Y_2) onto (X_2, Y_2) such that $hf=g$. Two PL-embeddings, f, g of the pair (X_1, Y_1) into (X_2, Y_2) such that $f| Y_1 = g| Y_1$ are *equivalent relative to* Y_2 if there exists a PL-homeomorphism h of the pair (X_2, Y_2) onto itself such that $hf=g$ and $h| Y_2$ is the identity map.

Let S^p, B^p denote the PL-p-sphere and PL-p-cell respectively. Zeeman [11] has shown that any two PL-embeddings of S^p into S^n and any two PL embeddings of $(B^p, \partial B^p$ (=boundary of B^p)) into $(B^n, \partial B^n)$ are equivalent provided $n-p \geqq 3$. Let X^{p-1} be a compact polyhedron and $v * X^{p-1}$ be the cone over X^{p-1}. Lickorish [7] generalized Zeeman's work to show that any two PL-embeddings of $(v * X^{p-1}, X^{p-1})$ into $(B^n, \partial B^n)$ are equivalent relative to ∂B^n provided $n-p \geqq 3$.

Let M^{n+1}, N^n be $(n+1)$- and n-dimensional closed PL-manifolds. In this paper, we investigate embeddings of $(v * N^n, N^n)$ into $(w * M^{n+1}, M^{n+1})$ which take v to w. If the embedding f of N^n into M^{n+1} induces isomorphisms of fundamental groups, we show that $w * M^{n+1}$ collapses to $f(v * N^n)$ for $n \geqq 5$. If, in addition, f and g are two locally unknotted embeddings which agree on N^n, then we can apply results of Akin [1] to obtain that f and g are equivalent relative to M. As an application of this result we obtain a covering concordance theorem.

We assume familiarity with either [5] or [12]. We express our gratitude to Henry Edwards, John Hollingsworth and Tom Kyrouz for their help and comments.

1. **Collapsing to embedded cones.** Suppose f: $(B^p, \partial B^p) \to (B^n, \partial B^n)$ is a PL-embedding of the pair. One of the essential steps in Zeeman's proof [11] was to show that B^n collapses to $f(B^p)$ if $n-p \geqq 3$. In [6], it was shown that B^n collapsed to $f(B^p)$ if $n=p+1 \geqq 6$. As noted in

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[6], collapsing in either $n=4$ or 5 implies the Schönflies Conjecture. We generalize as follows.

THEOREM 1. *Let M^{n+1}, N^n be $(n+1)$- and n-dimensional closed connected* PL-*manifolds, $n\geq 5$. Let f be a* PL-*embedding of the pair $(v*N^n, N^n)$ to $(w*M^{n+1}, M^{n+1})$ such that $f(v)=w$, $f(N^n)$ separates M^{n+1}, and f induces an isomorphism f_*: $\Pi_1(N^n)\to\Pi_1(M^{n+1})$; then $w*M^{n+1}$ collapses to $f(v*N^n)$.*

Let L, K be triangulations of $v*N^n$, $w*M^{n+1}$ respectively such that f: $L\to K$ is simplicial. Let $Q=\text{st}(a, K)$, which we may suppose does not intersect M^{n+1}, and let $R=N(f(v*N^n)\cap\text{cl}(w*M^{n+1}-Q), (K-\text{st}^\circ(a, K))'')$,—i.e. R is the second derived neighborhood of $f(v*N^n)\cap\text{cl}(w*M^{n+1}-Q)$ in the complement of the open star of a in K.

LEMMA 1. *$R\cup Q$ collapses to Q.*

PROOF. Note that R is a regular neighborhood of $f(v*N^n)\cap\text{cl}(w*M^{n+1}-Q)$ in the PL-manifold $\text{cl}(w*M^{n+1}-Q)$. Hence $R\cup Q$ collapses to $f(v*N^n)\cup Q$ which clearly collapses to Q.

The following lemma follows from Cohen [3].

LEMMA 2. *The triple $(R, R\cap M^{n+1}, R\cap Q)$ is* PL-*homeomorphic to the triple $(N^n\times I\times I, N^n\times I\times 0, N^n\times I\times 1)$ where $I=[0, 1]$.*

LEMMA 3. *Let S_i be a component of $S=\text{Cl}(w*M^{n+1}-(R\cup Q))$. The following homomorphisms induced by inclusion are isomorphisms.*

(1) $\Pi_1(S_1\cap M^{n+1})\to\Pi_1(M^{n+1})$

(2) $\Pi_1(f(v*N^n)\cap R)\to\Pi_1(R\cup S)$

(3) $\Pi_1(S_i)\to\Pi_1(R\cup S)$

(4) $\Pi_1(S_i\cap M^{n+1})\to\Pi_1(S_i)$

(5) $\Pi_1(S_i\cap Q)\to\Pi_1(S_i)$

PROOF. (1) Since $\Pi_1(fN^n)\to\Pi_1(M^{n+1})$ is an isomorphism, $\Pi_1(R\cap M^{n+1})\to\Pi_1(M^{n+1})$ is an isomorphism. From Lemma 2, $(R\cup S_i)\cap M^{n+1}$ is PL-homeomorphic to $S_i\cap M^{n+1}$; hence $\Pi_1(S_i\cap M^{n+1})\to\Pi_1(M^{n+1})$ is onto. Let g: $S^1\to S_i\cap M^{n+1}$ represent some element of $\Pi_1(S_1\cap M^{n+1})$ such that g can be extended to g: $D^2\to M^{n+1}$. We may asssme that g is a PL-embedding which is in general position with respect to fN^n. Hence $fN^n\cap gD^2$ is a finite collection of disjoint simple closed curves. Since $\Pi_1(fN^n)\to\Pi_1(M^{n+1})$ is an isomorphism, by the classical technique of "exchanging and popping" disks we can assume $fN^n\cap gD^2=\emptyset$ and hence $gD^2\subseteq S_i$.

(2) follows from the following commutative diagram:

$$\begin{array}{ccc} \Pi_1(fN^n) & \to & \Pi_1(f(v * N^n) \cap R) \\ \downarrow & & \downarrow \\ \Pi_1(M^{n+1}) & \to & \Pi_1(R \cup S) \end{array}$$

(3) follows from an argument similar to (1) and (4) follows from the following commutative diagram:

$$\begin{array}{ccc} \Pi_1(S_i \cap M^{n+1}) & \to & \Pi_1(S_i) \\ \downarrow & & \downarrow \\ \Pi_1(M^{n+1}) & \to & \Pi_1(R \cup S) \end{array}$$

The arguments for (5) follow a similar pattern.

LEMMA 4. *S is an h-cobordism between $S\cap Q$ and $S\cap M^{n+1}$.*

PROOF. Let $\tilde{X}$ denote the universal cover of X and let p: $\tilde{X}\to X$ be the natural map. If $Y\subset X$ and $\Pi_1 Y\to\Pi_1 X$ is an isomorphism, then $p^{-1}Y=\tilde{Y}$.

Consider the triple $(\tilde{R}\cup\tilde{S}, \tilde{R}\cup(S\cap Q)^\sim, ((R\cup S)\cap Q)^\sim)$. Clearly the relative homology groups $H_k(\tilde{R}\cup\tilde{S}, ((R\cup S)\cap Q)^\sim)$ and $H_k(\tilde{R}\cup(S\cap Q)^\sim, ((R\cup S)\cap Q)^\sim)$ are zero for all k. By exactness of the homology sequence of the triple, $H_k(\tilde{R}\cup\tilde{S}, \tilde{R}\cup(S\cap Q)^\sim)=0$ for all k. By excision, $H_k(\tilde{S}, (\tilde{R}\cap\tilde{S})\cup(S\cap Q)^\sim)=0$ for all k.

Consider the triple $(\tilde{S}, (\tilde{R}\cap\tilde{S})\cup(S\cap Q)^\sim, (S\cap Q)^\sim)$. From Lemma 2, it follows that $H_k(\tilde{S}, (S\cap Q)^\sim)=0$ for all k. Hence $\Pi_k(S, S\cap Q)=\Pi_k(\tilde{S}, (S\cap Q)^\sim)=0$ for all k by the Hurewicz Theorem and Lemma 3. Therefore $S\cap Q$ is a deformation retract of S. Similarly, $S\cap M$ is a deformation retract of S.

PROOF OF THEOREM 1. From [**9**], the Whitehead torsion $\tau(w * M^{n+1}, Q) = \tau(w * M^{n+1}, R\cup Q) + i_*\tau(R\cup Q, Q) = 0$ since $w * M^{n+1}$ collapses to Q. From Lemma 1, $\tau(R\cup Q, Q)=0$ and hence $i_*\tau(R\cup Q, Q) = 0$. Therefore $\tau(S, Q\cap S) = \tau(S, (R\cup Q)\cap S)=\tau(w * M^{n+1}, R\cup Q)=0$. By the s-cobordism Theorem [**10**], S is PL-homeomorphic to $(S\cap Q)\times I$. Hence S collapses to $S\cap Q$ and the theorem follows.

EXAMPLE. If, in Theorem 1, the condition that f_*: $\Pi_1(N^n)\to\Pi_1(M^{n+1})$ is not an isomorphism is not assumed, the conclusion may not be necessarily true. Consider the following example. Let (B^{n+2}, B^n) be a locally unknotted, knotted ball pair such that $(\partial B^{n+2}, \partial B^n)$ is unknotted. Such ball pairs exist for $n\geqq 1$. Let L be a regular neighborhood of B^n in B^{n+2} such that $L\cap\partial B^{n+2}$ is a regular neighborhood of ∂B^n in ∂B^{n+2}. Let $N=\partial(L\cap\partial B^{n+2})$, which is PL-homeomorphic to $S^{n-1}\times S^1$. Since L is a PL-$(n+2)$-cell, there is a PL-homeomorphism h: $L\to v*\partial L$. Consider h^{-1}: $(v*hN,\ hN)\to(B^{n+2},\ \partial B^{n+2})$. Note that $h^{-1}(v*hN)$ separates B^{n+2} into two components with one closure $h^{-1}(v*h(L\cap\partial B^{n+2}))$. If B^{n+2} collapses to $h^{-1}(v*hN)$, then B^{n+2} collapses to $h^{-1}(v*h(L\cap\partial B^{n+2}))$ which in turn collapses to B^n. Since $(B^{n+2},\ B^n)$ is locally unknotted, it follows from a proof similar to Zeeman's unknotting theorem [**11**] that $(B^{n+2},\ B^n)$ is unknotted. Hence B^{n+2} does not collapse to $h^{-1}(v*hN)$.

Let f, M^{n+1}, N^n be as in Theorem 1. f is said to be locally unknotted if for each point $p\in w*fN^n$, $p\neq w$, there exists a closed neighborhood U of p in $w*M^{n+1}$ such that $(U,\ U\cap fN^n)$ is an unknotted PL-ball pair.

The following theorem now follows from Akin [**1**] and Theorem 1.

THEOREM 2. *Let f, M^{n+1}, N^n be as in Theorem* 1 *and suppose that f is locally unknotted; then there exists a* PL-*homeomorphism h: $w*M^{n+1}\to w*M^{n+1}$ such that $h|M^{n+1}$ is the identity and $hf=\hat f$ where $\hat f$ is the standard conical extension of $f|N^n$.*

2. **Covering concordances.** Let M and N be compact PL-manifolds and let f: $N\to M$ be a PL-embedding; f is *proper* if $f^{-1}(\partial M)=\partial N$. A PL-*concordance* of N in M is a proper PL-embedding F: $N\times I\to M\times I$; an ambient PL-*concordance* of M is a proper PL-homeomorphism H: $M\times I\to M\times I$ which is the identity on $M\times 0$. *H covers F* if $H(F_0\times 1)=F$ where $F_0=F|N\times 0$.

Hudson [**4**] showed that any PL-concordance of N in M can be covered by an ambient concordance of M if codimension $N\leqq 3$. Lickorish and Siebenmann [**8**] extended these results in codimension $\leqq 3$ to the case when N is a polyhedron.

THEOREM 3. *Let M^{n+1}, N^n be $(n+1)$- and n-dimensional closed connected* PL-*manifolds, $n\geqq 5$. Let F be a locally unknotted concordance of N^n in M^{n+1} such that F induces an isomorphism F_*: $\Pi_1(N^n\times I)\to\Pi_1(M^{n+1}\times I)$; then F can be covered by an ambient concordance of M.*

PROOF. Let

$$C(N)=N\times I\cup a*(N\times 1),\qquad C(M)=M\times I\cup b*(M\times 1)$$

and extend F conically to an embedding F: $C(N)\to C(M)$. By Theorem 2, there exists a PL-homeomorphism h: $C(M)\to C(M)$ such that $h|M\times 0$ is the identity and $h\hat F=\hat F$, where $\hat F$ is the standard conical extension of $F|N\times 0$. Note, in particular, $\hat F(C(N))\cap(b*(M\times 1))=\hat F(a*(N\times 1))$. $b*(M\times 1)$ and $h(b*(M\times 1))$ are both relative regular neighborhoods of $\hat F(C(N))$ mod $\mathrm{Cl}(\hat F(C(N)-a*(N\times 1)))$ in the sense of Cohen [2]. By the uniqueness theorem [2], there is a PL-homeomorphism $k: C(M)\to C(M)$ such that $k|(M\times 0)\cup\hat{F}(C(N))$ is the identity and $k(h(b*(M\times 1)))=b*(M\times 1)$. Let $H=(kh)^{-1}|M\times I$.

EXAMPLE. If, in Theorem 3, the homomorphism F_*: $\Pi_1(N^n\times I)\to\Pi_1(M^{n+1}\times I)$ is not an isomorphism, the conclusion is not necessarily true. Consider the following example. Let (B^{n+2}, B^n) be a locally unknotted ball pair such that $(\partial B^{n+2}, \partial B^n)$ is knotted with $\Pi_1(\partial B^{n+2}-\partial B^n)$ not isomorphic to the integers. Let K be a triangulation of B^{n+2} containing $K|B^n$ and let L be the 2nd derived neighborhood of B^n in K. Let v be a vertex in the interior of B^n. Let $N_1=\partial(L\cap\partial B^{n+2})$, which is PL-homeomorphic to $S^{n-1}\times S^1$. Let $N_2=\partial(\mathrm{Link}(v, K'')\cap\mathrm{Cl}(L-st(v, K'')))$ which is PL-homeomorphic to $S^{n-1}\times S^1$. Then $N_1\cup\partial(L\cap[B^{n+1}-(\partial B^{n+1}\cup\mathrm{star}(v, K''))])\cup N_2$ is a concordance between N_1 and N_2 but there cannot be an ambient concordance since $\Pi_1(\mathrm{link}(v, K'')-\mathrm{link}(v, K''|B^n))$ is isomorphic to the integers.

REMARKS. Note that if one could prove Theorem 3 in the case when $n=3$, then all codimension one ball pairs unknot. All the theorems can be relativized to manifolds with boundary by considering proper embeddings with $n\geqq 6$.

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