UNKNOTTING IN CODIMENSION ONE

L. S. HUSCH[1]

Two PL (=Piecewise linear) embeddings f, g of a polyhedron X_1 into a polyhedron X_2 are *equivalent* if there exists a PL-homeomorphism h of X_2 onto X_2 such that $hf=g$. Let Y_i be a subpolyhedron of X_i, $i=1, 2$; a PL-map f of the pair (X_1, Y_1) into (X_2, Y_2) is a PL-map f: $X_1 \to X_2$ such that $f^{-1}(Y_2)=Y_1$. Two PL-embeddings, f, g, of the pair (X_1, Y_1) into (X_2, Y_2) are *equivalent* if there exists a PL-homeomorphism h of the pair (X_2, Y_2) onto (X_2, Y_2) such that $hf=g$. Two PL-embeddings, f, g of the pair (X_1, Y_1) into (X_2, Y_2) such that $f| Y_1 = g| Y_1$ are *equivalent relative to* Y_2 if there exists a PL-homeomorphism h of the pair (X_2, Y_2) onto itself such that $hf=g$ and $h| Y_2$ is the identity map.

Let S^p, B^p denote the PL-p-sphere and PL-p-cell respectively. Zeeman [**11**] has shown that any two PL-embeddings of S^p into S^n and any two PL embeddings of $(B^p, \partial B^p$ (=boundary of B^p)) into $(B^n, \partial B^n)$ are equivalent provided $n-p \geqq 3$. Let X^{p-1} be a compact polyhedron and $v * X^{p-1}$ be the cone over X^{p-1}. Lickorish [**7**] generalized Zeeman's work to show that any two PL-embeddings of $(v * X^{p-1}, X^{p-1})$ into $(B^n, \partial B^n)$ are equivalent relative to ∂B^n provided $n-p \geqq 3$.

Let M^{n+1}, N^n be $(n+1)$- and n-dimensional closed PL-manifolds. In this paper, we investigate embeddings of $(v * N^n, N^n)$ into $(w * M^{n+1}, M^{n+1})$ which take v to w. If the embedding f of N^n into M^{n+1} induces isomorphisms of fundamental groups, we show that $w * M^{n+1}$ collapses to $f(v * N^n)$ for $n \geqq 5$. If, in addition, f and g are two locally unknotted embeddings which agree on N^n, then we can apply results of Akin [**1**] to obtain that f and g are equivalent relative to M. As an application of this result we obtain a covering concordance theorem.

We assume familiarity with either [**5**] or [**12**]. We express our gratitude to Henry Edwards, John Hollingsworth and Tom Kyrouz for their help and comments.

1. **Collapsing to embedded cones.** Suppose f: $(B^p, \partial B^p) \to (B^n, \partial B^n)$ is a PL-embedding of the pair. One of the essential steps in Zeeman's proof [**11**] was to show that B^n collapses to $f(B^p)$ if $n-p \geqq 3$. In [**6**], it was shown that B^n collapsed to $f(B^p)$ if $n=p+1 \geqq 6$. As noted in

Received by the editors December 16, 1968.

[1] Research supported in part by National Science Foundation grant GP-8615.

[**6**], collapsing in either $n=4$ or 5 implies the Schönflies Conjecture. We generalize as follows.

THEOREM 1. *Let M^{n+1}, N^n be $(n+1)$- and n-dimensional closed connected* PL-*manifolds*, $n \geqq 5$. *Let f be a* PL-*embedding of the pair $(v * N^n, N^n)$ to $(w * M^{n+1}, M^{n+1})$ such that $f(v)=w$, $f(N^n)$ separates M^{n+1}, and f induces an isomorphism f_*: $\Pi_1(N^n) \to \Pi_1(M^{n+1})$; then $w * M^{n+1}$ collapses to $f(v * N^n)$.*

Let L, K be triangulations of $v * N^n$, $w * M^{n+1}$ respectively such that f: $L \to K$ is simplicial. Let $Q = \mathrm{st}(a, K)$, which we may suppose does not intersect M^{n+1}, and let $R = N(f(v * N^n) \cap \mathrm{cl}(w * M^{n+1} - Q), (K - \mathrm{st}^\circ(a, K))'')$,—i.e. R is the second derived neighborhood of $f(v * N^n) \cap \mathrm{cl}(w * M^{n+1} - Q)$ in the complement of the open star of a in K.

LEMMA 1. *$R \cup Q$ collapses to Q.*

PROOF. Note that R is a regular neighborhood of $f(v * N^n) \cap \mathrm{cl}(w * M^{n+1} - Q)$ in the PL-manifold $\mathrm{cl}(w * M^{n+1} - Q)$. Hence $R \cup Q$ collapses to $f(v * N^n) \cup Q$ which clearly collapses to Q.

The following lemma follows from Cohen [**3**].

LEMMA 2. *The triple $(R, R \cap M^{n+1}, R \cap Q)$ is* PL-*homeomorphic to the triple $(N^n \times I \times I, N^n \times I \times 0, N^n \times I \times 1)$ where $I = [0, 1]$.*

LEMMA 3. *Let S_i be a component of $S = \mathrm{Cl}(w * M^{n+1} - (R \cup Q))$. The following homomorphisms induced by inclusion are isomorphisms.*

(1) $\Pi_1(S_1 \cap M^{n+1}) \to \Pi_1(M^{n+1})$
(2) $\Pi_1(f(v * N^n) \cap R) \to \Pi_1(R \cup S)$
(3) $\Pi_1(S_i) \to \Pi_1(R \cup S)$
(4) $\Pi_1(S_i \cap M^{n+1}) \to \Pi_1(S_i)$
(5) $\Pi_1(S_i \cap Q) \to \Pi_1(S_i)$

PROOF. (1) Since $\Pi_1(fN^n) \to \Pi_1(M^{n+1})$ is an isomorphism, $\Pi_1(R \cap M^{n+1}) \to \Pi_1(M^{n+1})$ is an isomorphism. From Lemma 2, $(R \cup S_i) \cap M^{n+1}$ is PL-homeomorphic to $S_i \cap M^{n+1}$; hence $\Pi_1(S_i \cap M^{n+1}) \to \Pi_1(M^{n+1})$ is onto. Let g: $S^1 \to S_i \cap M^{n+1}$ represent some element of $\Pi_1(S_1 \cap M^{n+1})$ such that g can be extended to g: $D^2 \to M^{n+1}$. We may asssme that g is a PL-embedding which is in general position with respect to fN^n. Hence $fN^n \cap gD^2$ is a finite collection of disjoint simple closed curves. Since $\Pi_1(fN^n) \to \Pi_1(M^{n+1})$ is an isomorphism, by the classical technique of "exchanging and popping" disks we can assume $fN^n \cap gD^2 = \emptyset$ and hence $gD^2 \subseteq S_i$.

(2) follows from the following commutative diagram:

$$\begin{array}{ccc} \Pi_1(fN^n) & \to & \Pi_1(f(v * N^n) \cap R) \\ \downarrow & & \downarrow \\ \Pi_1(M^{n+1}) & \to & \Pi_1(R \cup S) \end{array}$$

(3) follows from an argument similar to (1) and (4) follows from the following commutative diagram:

$$\begin{array}{ccc} \Pi_1(S_i \cap M^{n+1}) & \to & \Pi_1(S_i) \\ \downarrow & & \downarrow \\ \Pi_1(M^{n+1}) & \to & \Pi_1(R \cup S) \end{array}$$

The arguments for (5) follow a similar pattern.

LEMMA 4. *S is an h-cobordism between $S \cap Q$ and $S \cap M^{n+1}$.*

PROOF. Let $\tilde{X}$ denote the universal cover of X and let $p: \tilde{X} \to X$ be the natural map. If $Y \subset X$ and $\Pi_1 Y \to \Pi_1 X$ is an isomorphism, then $p^{-1}Y = \tilde{Y}$.

Consider the triple $(\tilde{R} \cup \tilde{S}, \tilde{R} \cup (S \cap Q)^{\sim}, ((R \cup S) \cap Q)^{\sim})$. Clearly the relative homology groups $H_k(\tilde{R} \cup \tilde{S}, ((R \cup S) \cap Q)^{\sim})$ and $H_k(\tilde{R} \cup (S \cap Q)^{\sim}, ((R \cup S) \cap Q)^{\sim})$ are zero for all k. By exactness of the homology sequence of the triple, $H_k(\tilde{R} \cup \tilde{S}, \tilde{R} \cup (S \cap Q)^{\sim}) = 0$ for all k. By excision, $H_k(\tilde{S}, (\tilde{R} \cap \tilde{S}) \cup (S \cap Q)^{\sim}) = 0$ for all k.

Consider the triple $(\tilde{S}, (\tilde{R} \cap \tilde{S}) \cup (S \cap Q)^{\sim}, (S \cap Q)^{\sim})$. From Lemma 2, it follows that $H_k(\tilde{S}, (S \cap Q)^{\sim}) = 0$ for all k. Hence $\Pi_k(S, S \cap Q) = \Pi_k(\tilde{S}, (S \cap Q)^{\sim}) = 0$ for all k by the Hurewicz Theorem and Lemma 3. Therefore $S \cap Q$ is a deformation retract of S. Similarly, $S \cap M$ is a deformation retract of S.

PROOF OF THEOREM 1. From [9], the Whitehead torsion $\tau(w * M^{n+1}, Q) = \tau(w * M^{n+1}, R \cup Q) + i_* \tau(R \cup Q, Q) = 0$ since $w * M^{n+1}$ collapses to Q. From Lemma 1, $\tau(R \cup Q, Q) = 0$ and hence $i_* \tau(R \cup Q, Q) = 0$. Therefore $\tau(S, Q \cap S) = \tau(S, (R \cup Q) \cap S) = \tau(w * M^{n+1}, R \cup Q) = 0$. By the s-cobordism Theorem [10], S is PL-homeomorphic to $(S \cap Q) \times I$. Hence S collapses to $S \cap Q$ and the theorem follows.

EXAMPLE. If, in Theorem 1, the condition that $f_*: \Pi_1(N^n) \to \Pi_1(M^{n+1})$ is not an isomorphism is not assumed, the conclusion may not be necessarily true. Consider the following example. Let (B^{n+2}, B^n) be a locally unknotted, knotted ball pair such that $(\partial B^{n+2}, \partial B^n)$ is unknotted. Such ball pairs exist for $n \geqq 1$. Let L be a regular neighborhood of B^n in B^{n+2} such that $L \cap \partial B^{n+2}$ is a regular neighborhood of ∂B^n in ∂B^{n+2}. Let $N = \partial(L \cap \partial B^{n+2})$, which is PL-homeomorphic to $S^{n-1}\times S^1$. Since L is a PL-$(n+2)$-cell, there is a PL-homeomorphism h: $L\to v*\partial L$. Consider h^{-1}: $(v*hN, hN)\to(B^{n+2}, \partial B^{n+2})$. Note that $h^{-1}(v*hN)$ separates B^{n+2} into two components with one closure $h^{-1}(v*h(L\cap\partial B^{n+2}))$. If B^{n+2} collapses to $h^{-1}(v*hN)$, then B^{n+2} collapses to $h^{-1}(v*h(L\cap\partial B^{n+2}))$ which in turn collapses to B^n. Since (B^{n+2}, B^n) is locally unknotted, it follows from a proof similar to Zeeman's unknotting theorem [**11**] that (B^{n+2}, B^n) is unknotted. Hence B^{n+2} does not collapse to $h^{-1}(v*hN)$.

Let f, M^{n+1}, N^n be as in Theorem 1. f is said to be locally unknotted if for each point $p\in w*fN^n$, $p\neq w$, there exists a closed neighborhood U of p in $w*M^{n+1}$ such that $(U, U\cap fN^n)$ is an unknotted PL-ball pair.

The following theorem now follows from Akin [**1**] and Theorem 1.

THEOREM 2. *Let f, M^{n+1}, N^n be as in Theorem* 1 *and suppose that f is locally unknotted; then there exists a* PL-*homeomorphism h: $w*M^{n+1}\to w*M^{n+1}$ such that $h|M^{n+1}$ is the identity and $hf=\hat{f}$ where $\hat{f}$ is the standard conical extension of $f|N^n$.*

2. **Covering concordances.** Let M and N be compact PL-manifolds and let f: $N\to M$ be a PL-embedding; f is *proper* if $f^{-1}(\partial M)=\partial N$. A PL-*concordance* of N in M is a proper PL-embedding F: $N\times I\to M\times I$; an ambient PL-*concordance* of M is a proper PL-homeomorphism H: $M\times I\to M\times I$ which is the identity on $M\times 0$. *H covers F* if $H(F_0\times 1)=F$ where $F_0=F|N\times 0$.

Hudson [**4**] showed that any PL-concordance of N in M can be covered by an ambient concordance of M if codimension $N\leqq 3$. Lickorish and Siebenmann [**8**] extended these results in codimension $\leqq 3$ to the case when N is a polyhedron.

THEOREM 3. *Let M^{n+1}, N^n be $(n+1)$- and n-dimensional closed connected* PL-*manifolds, $n\geqq 5$. Let F be a locally unknotted concordance of N^n in M^{n+1} such that F induces an isomorphism F_*: $\Pi_1(N^n\times I)\to\Pi_1(M^{n+1}\times I)$; then F can be covered by an ambient concordance of M.*

PROOF. Let

$$C(N) = N\times I\cup a*(N\times 1), \qquad C(M) = M\times I\cup b*(M\times 1)$$

and extend F conically to an embedding F: $C(N)\to C(M)$. By Theorem 2, there exists a PL-homeomorphism h: $C(M)\to C(M)$ such that $h|M\times 0$ is the identity and $h\hat{F}=\hat{F}$, where $\hat{F}$ is the standard conical extension of $F|N\times 0$. Note, in particular, $\hat{F}(C(N))\cap(b*(M\times 1))=\hat{F}(a*(N\times 1))$. $b*(M\times 1)$ and $h(b*(M\times 1))$ are both relative regular neighborhoods of $\hat{F}(C(N))$ mod $\mathrm{Cl}(\hat{F}(C(N)-a*(N\times 1)))$ in the sense of Cohen [2]. By the uniqueness theorem [2], there is a PL-homeomorphism k: $C(M) \to C(M)$ such that $k|(M\times 0) \cup \hat{F}(C(N))$ is the identity and $k(h(b * (M\times 1))) = b * (M\times 1)$. Let $H = (kh)^{-1}| M\times I$.

EXAMPLE. If, in Theorem 3, the homomorphism F_*: $\Pi_1(N^n\times I) \to \Pi_1(M^{n+1}\times I)$ is not an isomorphism, the conclusion is not necessarily true. Consider the following example. Let (B^{n+2}, B^n) be a locally unknotted ball pair such that $(\partial B^{n+2}, \partial B^n)$ is knotted with $\Pi_1(\partial B^{n+2} - \partial B^n)$ not isomorphic to the integers. Let K be a triangulation of B^{n+2} containing $K|B^n$ and let L be the 2nd derived neighborhood of B^n in K. Let v be a vertex in the interior of B^n. Let $N_1 = \partial(L \cap \partial B^{n+2})$, which is PL-homeomorphic to $S^{n-1}\times S^1$. Let $N_2 = \partial(\mathrm{Link}(v, K'') \cap \mathrm{Cl}(L - st(v, K'')))$ which is PL-homeomorphic to $S^{n-1}\times S^1$. Then $N_1 \cup \partial(L \cap [B^{n+1} - (\partial B^{n+1} \cup \mathrm{star}(v, K''))]) \cup N_2$ is a concordance between N_1 and N_2 but there cannot be an ambient concordance since $\Pi_1(\mathrm{link}(v, K'') - \mathrm{link}(v, K''|B^n))$ is isomorphic to the integers.

REMARKS. Note that if one could prove Theorem 3 in the case when $n = 3$, then all codimension one ball pairs unknot. All the theorems can be relativized to manifolds with boundary by considering proper embeddings with $n \geqq 6$.

REFERENCES

1. E. Akin, *Manifold phenomena in the theory of polyhedra*, Ph.D. thesis, Princeton University, Princeton, N. J., 1968.

2. M. M. Cohen, *A general theory of relative regular neighborhoods*, Trans. Amer. Math. Soc. **136** (1969), 189–229.

3. M. M. Cohen and D. Sullivan, *On the regular neighborhood of a two-sided submanifold* (to appear).

4. J. F. P. Hudson, *Concordance and isotopy of* PL *embeddings*. Bull. Amer. Math. Soc. **72**(1966), 534–535.

5. ———, *Piecewise linear topology*. Vol. I, Mimeographed notes by J. L. Shaneson, University of Chicago, 1966/67.

6. L. S. Husch, *On collapsible ball pairs*, Illinois J. Math. **12**(1968), 414–420.

7. W. B. R. Lickorish, *The piecewise linear unknotting of cones*, Topology **4**(1965–66), 67–91.

8. W. B. R. Lickorish and L. C. Siebenmann, *Regular neighborhoods and the stable range*, Trans. Amer. Math. Soc. **139** (1969), 207–230.

9. J. W. Milnor, *Whitehead torsion*, Bull. Amer. Math. Soc. **72** (1966), 358–426.

10. J. B. Wagoner, Appendix I (P:S-*Cobordism*), Thesis, Princeton University, Princeton, N. J., 1966.

11. E. C. Zeeman, *Unknotting combinatorial balls*, Ann. of Math. **78**(1963), 501–526.

12. ———, *Seminar on combinatorial topology*, Mimeographed notes, Inst. Hautes Études Sci. Publ. Math., Paris, 1963.

UNIVERSITY OF GEORGIA