The main result of this paper is the following theorem.

**Theorem.** Let $M^n$ denote a nonbounded combinatorial manifold rectilinearly imbedded in some Euclidean space. If $M^n$ possesses a transverse field, then $M^n$ has a normal microbundle.

We assume that the reader is familiar with the theory of piecewise linear microbundles, which was developed by J. Milnor (see [5] or [6]), as well as familiarity with the theory of $C^1$-complexes, developed by J. H. C. Whitehead, (see [7] or [10]).

0. **Convention.** Throughout this paper $M^n$ will denote a nonbounded combinatorial manifold rectilinearly imbedded in some Euclidean space.

1. **Definition.** Let $s$ denote a simplex (open) of $M$. By $P(s)$ we will denote the subset (possibly empty) in $G_{n,p}$ (the Grassmannian manifold of all $p$-planes through the origin in $R^{n+p}$) consisting of all $p$-planes $P$, such that, if $H$ is the orthogonal $n$-plane to $P$ in $R^{n+p}$, then the orthogonal projection $q: R^{n+p} \rightarrow H$, restricted to $St(s, M)^-$ (the closed star of $s$ in $M$) is a homeomorphism carrying $St(s, M)^-$ onto an open set in $H$. If the $p$-plane $P$ belongs to $P(s)$, then $P$ is said to be transversal to $M$ at $m$, where $m$ is any point of $s$. A continuous map $g: M \rightarrow G_{n,p}$ is a transverse field, if the set $g(s)$ is contained in the set $P(s)$, for every simplex $s$ of $M$.

2. **Remark.** If the simplex $t$ is a face of the simplex $s$, then since $St(t, M)^-$ is contained in $St(s, M)^-$, we have the fact that the set $P(t)$ is contained in the set $P(s)$.

3. **Lemma.** Let $g: M^n \rightarrow G_{n,p}$ be a transverse field. Without loss of generality we may assume that the map, $g$, is piecewise smooth.

**Proof.** Suppose we have a continuous transverse field, $h: M^n \rightarrow G_{n,p}$. By Remark 2 we see that $h(s)$ is contained in $P(s)$. The reader can verify that the sets $P(s)$ are open in $G_{n,p}$ (or see Proposition 3.7 of [9]). Thus, since $h(s)$ is compact, we see that there exists a positive continuous function $\delta$ mapping $M$ to the positive reals, such that, if

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4. **Lemma.** Given a transverse field $g: M \rightarrow G_{n,p}$. Let $N$ denote the subset of $M \times R^{n+p}$ consisting of pairs $(x, v)$ such that the vector $v$ belongs to the $p$-plane $g(x)$. Then $N$ is a vector bundle with base $M$, which we shall call a normal bundle of the imbedding of $M$ in $R^{n+p}$.

**Proof.** Consider the $p$-dimensional vector bundle $G$ consisting of pairs $(P, v)$ in the space $G_{n,p} \times R^{n+p}$ such that the vector $v$ belongs to the $p$-plane $P$. The set $N$ then, is just the bundle induced by the map $g$.

5. **Remark.** Let $g: M \rightarrow G_{n,p}$ be a piecewise smooth transverse field. (This is no loss in generality by Lemma 3.) The map $g$ induces a map $f: N \rightarrow R^{n+p}$ defined by $f(x, v) = x + v$. Let $m$ be a point of $M$, then since a vector bundle over a contractible set is trivial, let $h: St(m, M)^{-} \times R^{p} \rightarrow N$ denote a trivialization of $N$ over $St(m, M)^{-}$. The map $h$ may be chosen such that the map $k: St(m, M)^{-} \times R^{p} \rightarrow R^{n+p}$ induced by $f$ is piecewise smooth. (We give $St(m, M)^{-} \times R^{p}$ a simplicial structure.) Let $e_1, \ldots, e_p$ denote an orthonormal basis of $R^{p}$, and set $h(x, e_i) = (x, v_i(x))$, a point of $N$. Then for a point $w$ in $R^{p}$ such that $w = \sum c_i e_i$, we have that $k(x, w) = x + \sum c_i v_i(x)$. Finally we define a new map $k': St(m, M)^{-} \times R^{p} \rightarrow R^{n+p}$ by $k'(x, w) = x + \sum c_i v_i(m)$.

6. **Lemma.** The map $k': St(m, M)^{-} \times R^{p} \rightarrow R^{n+p}$ defined in the previous remark is a piecewise smooth imbedding.

**Proof.** Let $X = St(m, M)^{-} \times R^{p}$ and $b = (x, w)$. We must prove that:

1. $dk'_b : St(b, X)^{-} \rightarrow R^{n+p}$ is one-to-one for each point $b$ of $X$,
2. $k'$ is a homeomorphism.

**Proof of (1).** Recall that $dk'_b (z) = Dk'(b) \cdot (z - b)$ (where $Dk'(b)$ is the Jacobian of the map $k'$ with respect to a simplex containing $b$. Though the Jacobian depends upon the simplex chosen, the map $dk'_b$ does not). To prove that the map $dk'_b$ is one-to-one, it is obviously sufficient to prove that the Jacobian $Dk'(b)$ is nonsingular. Now the $n+p \times n+p$ matrix $Dk'(b) = (C; v_1(m); \cdots; v_p(m))$, where
the \( n \) columns of the \( n+p \times n \) matrix \( C \) span the \( n \)-plane determined by an \( n \)-simplex in \( \text{St}(m, M)^- \) containing the point \( x \). Since the vectors \( v_i(m), i = 1, \ldots, p \) span the \( p \)-plane \( g(m) \) and are not in the \( n \)-plane determined by any simplex of \( \text{St}(m, M)^- \), the Jacobian \( Dk'(b) \) is nonsingular.

**Proof of (2).** It is easily seen that it suffices to prove that the map \( k' \) is one-to-one. If \( k' \) is not one-to-one, then there exist pairs \( (x, w) \) and \( (y, u) \), where \( w = \sum c_i e_i \) and \( u = \sum d_i e_i \), such that \( x + \sum c_i v_i(m) = y + \sum d_i v_i(m) \). This implies that the vector \( x - y \) belongs to the \( p \)-plane \( g(m) \), and hence that the projection of \( \text{St}(m, M)^- \) to the orthogonal \( n \)-plane to \( g(m) \) is not one-to-one. This contradicts the choice of \( g(m) \).

7. **Lemma.** Given a point \( m \) of \( M \), there exists a finite subcomplex \( Y \) (depending on the point \( m \)) of some subdivision of \( \text{St}(m, M)^- \times \mathbb{R}^p \), which contains a neighborhood of \( (m, 0) \) on which the map \( k \) (defined in Remark 5) is a piecewise smooth imbedding.

**Proof.** Let \( Z \) be a finite subcomplex of \( \text{St}(m, M)^- \times \mathbb{R}^p \) consisting of pairs \( (x, w) \) such that \( \|w\| \leq 1 \). By Theorem 8.8 of [7], and the previous lemma, there is a \( \delta > 0 \), such that any \( \delta \)-approximation to the map \( k' \) is a piecewise smooth imbedding, for any subcomplex \( Y \) contained in an arbitrary subdivision of \( Z \). To insure that the map \( k \) is the desired \( \delta \)-approximation, we choose \( Y \) as follows. First, for \( w = \sum c_i e_i \) in \( \mathbb{R}^p \), let \( w(x) = \sum c_i v_i(x) \) in \( \mathbb{R}^{n+p} \). Now we have that \( \|k(x, w) - k'(x, w)\| = \|w(x) - w(m)\| \). Thus on \( Z \), since \( \|w\| \leq 1 \), we may choose the point \( x \) sufficiently close to \( m \), to insure that \( \|w(x) - w(m)\| < \delta \). We next wish to choose \( b = (x, w) \) sufficiently close to \( (m, 0) \) so that \( \|Dk_b(z) - Dk'_b(z)\| < \delta \|z - b\| \), or equivalently so that \( \|Dk(x, w) - Dk'(x, w)\| < c \delta \). (The norm of a matrix is the absolute value of its maximum entry; \( c = (n+p)^{3/2} \).

Now \( Dk(x, w) = (C + c_1 B_1(x) + \cdots + c_p B_p(x); v_1(x); \cdots; v_p(x)) \) where the \( n \) columns of the \( n+p \times n \) matrix \( C \) span the \( n \)-plane determined by an \( n \)-simplex in \( \text{St}(m, M)^- \) containing \( x \), and the entries of the \( n+p \times n \) matrices \( B_i(x) \) are bounded. Since \( Dk'(x, w) = (C; v_1(m); \cdots; v_p(m)) \) we have that \( \|Dk(x, w) - Dk'(x, w)\| = \|(c_1 B_1(x) + \cdots + c_p B_p(x); v_1(x) - v_1(m); \cdots; v_p(x) - v_p(m))\| \). Thus choose \( \|w\| \) small enough, and \( x \) sufficiently close to \( m \) to insure the desired inequality. (Of course the Jacobians chosen depend upon the \( n+p \) simplex chosen to contain the point \( x, w \), but there are only a finite number of such simplices.) Let \( U \) be the neighborhood of \( (m, 0) \) so determined to insure the above inequalities. Subdivide the complex \( Z \) sufficiently fine to obtain the desired complex \( Y \) contained in \( U \).
8. **Corollary.** There is a neighborhood of the zero section of the normal bundle \( N \) of \( M \) on which the map \( f: N \to \mathbb{R}^{n+p} \) induced by the piecewise smooth transverse field \( g: M \to G_{n+p} \) is a homeomorphism which maps this neighborhood onto an open subset of \( \mathbb{R}^{n+p} \).

**Proof.** By Lemma 5.7 of [7], it is sufficient to prove that each point \((m, 0)\) in the zero section of \( N \) has a neighborhood which \( f \) maps homeomorphically onto an open set of \( \mathbb{R}^{n+p} \). But this holds by Lemma 7, the desired neighborhood being the interior of the complex \( Y \). The image is open, for if \( m \) belongs to the simplex \( s \), the fact that \( P(s) \) is nonempty implies that \( St(m, M) \times R^p \), which equals \( St(s, M) \times R^p \). is homeomorphic with a subset of Euclidean \( n+p \) space. Thus Brouwer's theorem on the invariance of domain [3] implies that the image is open.

Let \( V \) denote the neighborhood of the zero section of the normal bundle \( N \) which by the previous corollary, \( f \) maps homeomorphically onto an open set \( f(V) \) in \( \mathbb{R}^{n+p} \). We shall give \( f(V) \) a microbundle structure so that it is a normal microbundle of \( M \). In order to do this we need the result that every vector bundle has a microbundle structure (see [4] or [8]). In fact, we use not only the result, but also the proof of that result as presented in [8].

9. **Theorem.** Let \( M^n \) denote a nonbounded combinatorial manifold rectilinearly imbedded in some Euclidean space. If \( M^n \) possesses a transverse field, then \( M^n \) has a normal microbundle.

**Proof.** Let \( e: K \to N \) be the triangulation of the normal bundle \( N \) given in [8]. The simplicial complex \( K \) is thus a microbundle over \( M \). Let \( L = e^{-1}(V) \). According to [1, p. 143] we may choose a rectilinear triangulation of \( L \) such that the inclusion map of \( L \) into \( K \) is piecewise linear. The map \( fe: L \to \mathbb{R}^{n+p} \) is a piecewise smooth imbedding. This is due to the fact that in constructing the triangulation \((e, K)\) of the normal bundle \( N \), we may, as in [8], start with the piecewise smooth imbeddings \( h: St(m, M) \times R^p \to N \), where \( m \) is a vertex of \( M \). The maps \( h \) are then changed slightly to maps \( h' \) which we choose sufficiently close to the maps \( h \) to insure that the map \( fe \), which locally is the map \( fh' \), is a piecewise smooth imbedding. (We leave the details to the reader, noting however, that for a detailed proof, Lemma 7 is essential.) Now let \( d: L \to \mathbb{R}^{n+p} \) denote a piecewise linear approximation to the map \( fe \), sufficiently close to insure that it is a piecewise smooth imbedding, and that \( d(L) = fe(L) = f(V) \), an open set in \( \mathbb{R}^{n+p} \). To show that \( d(L) \) is a microbundle with base \( M \), it is sufficient to show that \( L \) is a microbundle with base \( M \). But any open neighborhood of the zero section of a microbundle is a microbundle over the
base. (In the definition of a microbundle we may replace $R^n$ by any
open ball about the origin, since the two are piecewise linearly homeo-
morphic.) Thus $d(L)$ is a normal microbundle for $M$, the piecewise
linear map $j: d(L) \to M$ is that induced by the piecewise linear projection $K \to M$.

We now prove a corollary to this theorem. Since this corollary is
independent of the global codimension of the imbedding of $M$ it differs
from results of A. Haefliger and C. T. C. Wall [2].

10. **Definition.** The imbedding of $M^n$ in $R^{n+p}$ is *locally normal* if for
each vertex $m$ of $M$ the set $P(m)$ is nonempty. Hence by Remark 2
the set $P(s)$ is nonempty for each simplex $s$.

11. **Definition.** The imbedding of $M^n$ in $R^{n+p}$ is of *local codimen-
sion $k$* if for each vertex $m$ of $M$ there is an $n+k$ plane $J$ (depending
on $m$) such that $St(m, M)$ is contained in some translation of $J$. (This
suggestive terminology of local codimension is due to H. S. Davis.)

12. **Corollary to Theorem 9.** Let $M^n$ denote a nonbounded combi-
natorial manifold rectilinearly imbedded in some Euclidean space. If
the imbedding is locally normal and of local codimension 1, then $M^n$
possesses a normal microbundle.

**Proof.** The principal result of [9, Theorem 1.8], states that the
hypothesis implies that $M^n$ has a transverse field. Hence by Theorem
9, $M^n$ possesses a normal microbundle.

**References**

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