TRANSVERSE FIELD IMPLIES NORMAL MICROBUNDLE

H. PUTZ

The main result of this paper is the following theorem.

THEOREM. *Let M^n denote a nonbounded combinatorial manifold rectilinearly imbedded in some Euclidean space. If M^n possesses a transverse field, then M^n has a normal microbundle.*

We assume that the reader is familiar with the theory of piecewise linear microbundles, which was developed by J. Milnor (see [**5**] or [**6**]), as well as familiarity with the theory of C^1-complexes, developed by J. H. C. Whitehead, (see [**7**] or [**10**]).

0. **Convention.** Throughout this paper M^n will denote a nonbounded combinatorial manifold rectilinearly imbedded in some Euclidean space.

1. **Definition.** Let s denote a simplex (open) of M. By $P(s)$ we will denote the subset (possibly empty) in $G_{n,p}$ (the Grassmannian manifold of all p-planes through the origin in R^{n+p}) consisting of all p-planes P, such that, if H is the orthogonal n-plane to P in R^{n+p}, then the orthogonal projection $q: R^{n+p} \rightarrow H$, restricted to $\mathrm{St}(s, M)^-$ (the closed star of s in M) is a homeomorphism carrying $\mathrm{St}(s, M)$ onto an open set in H. If the p-plane P belongs to $P(s)$, then P is said to be *transversal* to M at m, where m is any point of s. A continuous map $g: M \rightarrow G_{n,p}$ is a *transverse field*, if the set $g(s)$ is contained in the set $P(s)$, for every simplex s of M.

2. **Remark.** If the simplex t is a face of the simplex s, then since $\mathrm{St}(s, M)^-$ is contained in $\mathrm{St}(t, M)^-$, we have the fact that the set $P(t)$ is contained in the set $P(s)$.

3. **Lemma.** *Let $g: M^n \rightarrow G_{n,p}$ be a transverse field. Without loss of generality we may assume that the map, g, is piecewise smooth.*

PROOF. Suppose we have a continous transverse field, $h: M^n \rightarrow G_{n,p}$. By Remark 2 we see that $h(\bar{s})$ is contained in $P(s)$. The reader can verify that the sets $P(s)$ are open in $G_{n,p}$ (or see Proposition 3.7 of [**9**]). Thus, since $h(\bar{s})$ is compact, we see that there exists a positive continuous function δ mapping M to the positive reals, such that, if

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$g: M \to G_{n,p}$, and $\|g(x)-h(x)\| < \delta(x)$ (where $\| \ \|$ denotes a metric on $G_{n,p}$), then $g(s)$ is contained in $P(s)$. Now let V denote the second regular neighborhood of M in R^{n+p}, and let $r: V \to M$ denote the retraction of this open set onto M. Thus we have a continuous map $hr: V \to G_{n,p}$ defined on the smooth manifold V. Let $g: V \to G_{n,p}$ be a smooth map such that its restriction to M is pointwise within $\delta(x)$ of the map h. (Such a map exists by the proof of Theorem 4.2 of [**4**].) If we let the restriction map also be denoted by g, we have the proof of the lemma.

4. **Lemma.** *Given a transverse field $g: M \to G_{n,p}$. Let N denote the subset of $M \times R^{n+p}$ consisting of pairs (x, v) such that the vector v belongs to the p-plane $g(x)$. Then N is a vector bundle with base M, which we shall call a normal bundle of the imbedding of M in R^{n+p}.*

PROOF. Consider the p-dimensional vector bundle G consisting of pairs (P, v) in the space $G_{n,p} \times R^{n+p}$ such that the vector v belongs to the p-plane P. The set N then, is just the bundle induced by the map g.

5. **Remark.** Let $g: M \to G_{n,p}$ be a piecewise smooth transverse field. (This is no loss in generality by Lemma 3.) The map g induces a map $f: N \to R^{n+p}$ defined by $f(x, v) = x + v$. Let m be a point of M, then since a vector bundle over a contractible set is trivial, let $h: \mathrm{St}(m, M)^- \times R^p \to N$ denote a trivialization of N over $\mathrm{St}(m, M)^-$. The map h may be chosen such that the map $k: \mathrm{St}(m, M)^- \times R^p \to R^{n+p}$ induced by f is piecewise smooth. (We give $\mathrm{St}(m, M)^- \times R^p$ a simplicial structure.) Let $e_1, \cdots, e_p$ denote an orthonormal basis of R^p, and set $h(x, e_i) = (x, v_i(x))$, a point of N. Then for a point w in R^p such that $w = \sum c_i e_i$, we have that $k(x, w) = x + \sum c_i v_i(x)$. Finally we define a new map $k': \mathrm{St}(m, M)^- \times R^p \to R^{n+p}$ by $k'(x, w) = x + \sum c_i v_i(m)$.

6. **Lemma.** *The map $k': \mathrm{St}(m, M)^- \times R^p \to R^{n+p}$ defined in the previous remark is a piecewise smooth imbedding.*

PROOF. Let $X = \mathrm{St}(m, M)^- \times R^p$ and $b = (x, w)$. We must prove that: (1) $dk'_b : \mathrm{St}(b, X)^- \to R^{n+p}$ is one-to-one for each point b of X, (2) k' is a homeomorphism.

PROOF OF (1). Recall that $dk'_b(z) = Dk'(b) \cdot (z - b)$ (where $Dk'(b)$ is the Jacobian of the map k' with respect to a simplex containing b. Though the Jacobian depends upon the simplex chosen, the map dk'_b does not). To prove that the map dk'_b is one-to-one, it is obviously sufficient to prove that the Jacobian $Dk'(b)$ is nonsingular. Now the $n+p \times n+p$ matrix $Dk'(b) = (C; v_1(m); \cdots; v_p(m))$, where the n columns of the $n+p\times n$ matrix C span the n-plane determined by an n-simplex in $\mathrm{St}(m, M)^-$ containing the point x. Since the vectors $v_i(m)$, $i=1, \cdots, p$ span the p-plane $g(m)$ and are not in the n-plane determined by any simplex of $\mathrm{St}(m, M)^-$, the Jacobian $Dk'(b)$ is nonsingular.

PROOF OF (2). It is easily seen that it suffices to prove that the map k' is one-to-one. If k' is not one-to-one, then there exist pairs (x, w) and (y, u), where $w=\sum c_ie_i$ and $u=\sum d_ie_i$, such that $x+\sum c_iv_i(m) =y+\sum d_iv_i(m)$. This implies that the vector $x-y$ belongs to the p-plane $g(m)$, and hence that the projection of $\mathrm{St}(m, M)^-$ to the orthogonal n-plane to $g(m)$ is not one-to-one. This contradicts the choice of $g(m)$.

7. **Lemma.** *Given a point m of M, there exists a finite subcomplex Y (depending on the point m) of some subdivision of* $\mathrm{St}(m, M)^-\times R^p$, *which contains a neighborhood of* $(m, 0)$ *on which the map k (defined in Remark* 5) *is a piecewise smooth imbedding.*

PROOF. Let Z be a finite subcomplex of $\mathrm{St}(m, M)^-\times R^p$ consisting of pairs (x, w) *such that* $\|w\|\leqq 1$. By Theorem 8.8 of [**7**], and the previous lemma, there is a $\delta>0$, such that any δ-approximation to the map k' is a piecewise smooth imbedding, for any subcomplex Y contained in an arbitrary subdivision of Z. To insure that the map k is the desired δ-approximation, we choose Y as follows. First, for $w=\sum c_ie_i$ in R^p, let $w(x)=\sum c_iv_i(x)$ in R^{n+p}. Now we have that $\|k(x, w)-k'(x, w)\|=\|w(x)-w(m)\|$. Thus on Z, since $\|w\|\leqq 1$, we may choose the point x sufficiently close to m, to insure that $\|w(x)-w(m)\|<\delta$. We next wish to choose $b=(x, w)$ sufficiently close to $(m, 0)$ so that $\|dk_b(z)-dk_b'(z)\|<\delta\|z-b\|$, or equivalently so that $\|Dk(x, w)-Dk'(x, w)\|<c\delta$. (The norm of a matrix is the absolute value of its maximum entry; $c=(n+p)^{3/2}$.)

Now $Dk(x, w)=(C+c_1B_1(x)+\cdots+c_pB_p(x);\ v_1(x);\cdots;\ v_p(x))$ where the n columns of the $n+p\times n$ matrix C span the n-plane determined by an n-simplex in $\mathrm{St}(m, M)^-$ containing x, and the entries of the $n+p\times n$ matrices $B_i(x)$ are bounded. Since $Dk'(x, w) =(C;\ v_1(m);\cdots;\ v_p(m))$ we have that $\|Dk(x, w)-Dk'(x, w)\| =\|(c_1B_1(x)+\cdots+c_pB_p(x);\ v_1(x)-v_1(m);\cdots;\ v_p(x)-v_p(m))\|$. Thus choose $\|w\|$ small enough, and x sufficiently close to m to insure the desired inequality. (Of course the Jacobians chosen depend upon the $n+p$ simplex chosen to contain the point (x, w), but there are only a finite number of such simplices.) Let U be the neighborhood of $(m, 0)$ so determined to insure the above inequalities. Subdivide the complex Z sufficiently fine to obtain the desired complex Y contained in U.

8. **Corollary.** *There is a neighborhood of the zero section of the normal bundle N of M on which the map $f\colon N \to R^{n+p}$ induced by the piecewise smooth transverse field $g\colon M \to G_{n,p}$ is a homeomorphism which maps this neighborhood onto an open subset of R^{n+p}.*

Proof. By Lemma 5.7 of [7], it is sufficient to prove that each point $(m, 0)$ in the zero section of N has a neighborhood which f maps homeomorphically onto an open set of R^{n+p}. But this holds by Lemma 7, the desired neighborhood being the interior of the complex Y. The image is open, for if m belongs to the simplex s, the fact that $P(s)$ is nonempty implies that $\mathrm{St}(m, M)^{-} \times R^p$, which equals $\mathrm{St}(s, M)^{-} \times R^p$. is homeomorphic with a subset of Euclidean $n+p$ space. Thus Brouwer's theorem on the invariance of domain [3] implies that the image is open.

Let V denote the neighborhood of the zero section of the normal bundle N which by the previous corollary, f maps homeomorphically onto an open set $f(V)$ in R^{n+p}. We shall give $f(V)$ a microbundle structure so that it is a normal microbundle of M. In order to do this we need the result that every vector bundle has a microbundle structure (see [4] or [8]). In fact, we use not only the result, but also the proof of that result as presented in [8].

9. **Theorem.** *Let M^n denote a nonbounded combinatorial manifold rectilinearly imbedded in some Euclidean space. If M^n possesses a transverse field, then M^n has a normal microbundle.*

Proof. Let $e\colon K \to N$ be the triangulation of the normal bundle N given in [8]. The simplicial complex K is thus a microbundle over M. Let $L = e^{-1}(V)$. According to [1, p. 143] we may choose a rectilinear triangulation of L such that the incluslon map of L into K is piecewise linear. The map $fe\colon L \to R^{n+p}$ is a piecewise smooth imbedding. This is due to the fact that in constructing the triangulation (e, K) of the normal bundle N, we may, as in [8], start with the piecewise smooth imbeddings $h\colon \mathrm{St}(m, M) \times R^p \to N$, where m is a vertex of M. The maps h are then changed slightly to maps h' which we choose sufficiently close to the maps h to insure that the map fe, which locally is the map fh', is a piecewise smooth imbedding. (We leave the details to the reader, noting however, that for a detailed proof, Lemma 7 is essential.) Now let $d\colon L \to R^{n+p}$ denote a piecewise linear approximation to the map fe, sufficiently close to insure that it is a piecewise smooth imbedding, and that $d(L) = fe(L) = f(V)$, an open set in R^{n+p}. To show that $d(L)$ is a microbundle with base M, it is sufficient to show that L is a microbundle with base M. But any open neighborhood of the zero section of a microbundle is a microbundle over the base. (In the definition of a microbundle we may replace R^q by any open ball about the origin, since the two are piecewise linearly homeomorphic.) Thus $d(L)$ is a normal microbundle for M, the piecewise linear map $j: d(L) \to M$ is that induced by the piecewise linear projection $K \to M$.

We now prove a corollary to this theorem. Since this corollary is independent of the global codimension of the imbedding of M it differs from results of A. Haefliger and C. T. C. Wall [2].

10. **Definition.** The imbedding of M^n in R^{n+p} is *locally normal* if for each vertex m of M the set $P(m)$ is nonempty. Hence by Remark 2 the set $P(s)$ is nonempty for each simplex s.

11. **Definition.** The imbedding of M^n in R^{n+p} is of *local codimension k* if for each vertex m of M there is an $n+k$ plane J (depending on m) such that $\mathrm{St}(m, M)$ is contained in some translation of J. (This suggestive terminology of local codimension is due to H. S. Davis.)

12. **Corollary to Theorem 9.** *Let M^n denote a nonbounded combinatorial manifold rectilinearly imbedded in some Euclidean space. If the imbedding is locally normal and of local codimension* 1, *then M^n possesses a normal microbundle.*

Proof. The principal result of [**9**, Theorem 1.8], states that the hypothesis implies that M^n has a transverse field. Hence by Theorem 9, M^n possesses a normal microbundle.

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Temple University