INEQUALITIES FOR THE EIGENVALUES OF POWERS OF FUNCTIONS

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In this paper, we give generalizations of a result of A. M. Fink. Specifically, we consider the Sturm-Liouville system

(1) $$(p(x)u')' + \lambda\rho(x)u = 0, \qquad x \in (0, l),$$

(2) $$p(0)u'(0) - h_0u(0) = 0, \qquad p(l)u'(l) + h_1u(l) = 0,$$

where p, ρ are integrable functions on $[0, l]$ with p' continuous and ρ positive in a subinterval of $[0, l]$. Furthermore, h_0, h_1 are nonnegative numbers with h_0, h_1 approaching ∞ corresponding to the boundary conditions $u(0) = 0$, $u(l) = 0$, respectively. We will be concerned with the dependence of the positive eigenvalues of the system (1), (2) on the function ρ. We denote them accordingly by $0 < \lambda_1[\rho] < \lambda_2[\rho] < \cdots$. Fink shows that in the case where (1), (2) reduces to

$$u'' + \lambda\rho(x)u = 0, \qquad x \in [0, l],$$
$$u(0) = u(l) = 0,$$

the inequality

(3) $$\lambda_1[\rho^2] \leqq (l\lambda_1[\rho]/\pi)^2$$

holds [**1**]. We prove the following

THEOREM 1. *The positive eigenvalues of* (1), (2) *satisfy the functional inequality*

(4) $$\lambda_{m+n-1}[\rho] \geqq (\lambda_m[\,|\rho_1|^\alpha])^{1/\alpha}(\lambda_n[\,|\rho_2|^\beta])^{1/\beta}$$

where $\rho = \rho_1 \cdot \rho_2$ *and* $1/\alpha + 1/\beta = 1$.

When $\rho_2 \equiv 1$ on $[0, l]$, $n = m = 1$, $k = 1$, $\alpha = \beta = 2$, (4) reduces to Fink's inequality (3). Furthermore, the proof given here generalizes trivially to higher dimensional problems.

PROOF OF THEOREM. We make use of the Rayleigh quotient

(5) $$R[\rho, u] = \frac{\int_0^l u'^2dx + h_0u^2(0) + h_1u^2(l)}{\int_0^l \rho u^2dx}$$

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and the max-min characterization of the eigenvalues, i.e.,

$$\lambda_{n+1}[\rho] = \max_{[v_1, \cdots, v_n]} d[v_1, \cdots, v_n], \tag{6}$$

where $[v_1, \cdots, v_n]$ is the subspace spanned by n smooth functions $v_1, \cdots, v_n$ and

$$d[v_1, \cdots, v_n] = \min R[\rho, u], \tag{7}$$

where the minimum is taken over all smooth functions satisfying the conditions

$$(u, v_k) = \int_0^l \rho u v_k dx = 0$$

$(k=1, \cdots, n)$ and $\int_0^l \rho u^2 dx \geqq 0$ (see [2, p. 214]). By Hölder's inequality, we have

$$\begin{aligned} \int_0^l \rho u^2 dx &= \int_0^l (\rho_1 u^{2/\alpha})(\rho_2 u^{2/\beta}) dx \\ &\leqq \left(\int_0^l |\rho_1|^\alpha u^2 dx\right)^{1/\alpha} \left(\int_0^l |\rho_2|^\beta u^2 dx\right)^{1/\beta} \end{aligned} \tag{8}$$

with $1/\alpha + 1/\beta = 1$.

Let $U = [u_1, u_2, \cdots, u_{m-1}]$ where $\{u_k\}_{k=1}^{m-1}$ is the set of eigenfunctions corresponding to $\{\lambda_k[|\rho_1|]\}_{k=1}^{n-1}$ and let $V = [v_1, v_2, \cdots, v_{n-1}]$ where $\{v_k\}_{k=1}^{n-1}$ is the set corresponding to $\{\lambda_k[|\rho_2|]\}_{k=1}^{n-1}$.

If $W = [u_1, \cdots, u_{m-1}, v_1, \cdots, v_{n-1}]$, then (6) and (7) yield

$$\lambda_{m+n-1}[\rho] \geqq \min R[\rho_1 u],$$

where the minimum is taken over the set $\{u: u \in C', u \perp W, \int_0^l \rho u^2 dx \geqq 0\}$. By (8) and another application of the max-min principle we have

$$\begin{aligned} \lambda_{m+n-1}[\rho] &\geqq \min_{u \perp W} (R[|\rho_1|^\alpha, u])^{1/\alpha} (R[|\rho_2|^\beta, u])^{1/\beta} \\ &\geqq (\min_{u \perp U} R[|\rho_1|^\alpha, u])^{1/\alpha} (\min_{u \perp V} R[|\rho_2|^\beta, u])^{1/\beta} \\ &= (\lambda_{m-1}[|\rho_1|^\alpha])^{1/\alpha} (\lambda_{n-1}[|\rho_2|^\beta])^{1/\beta}. \end{aligned}$$

Noting that the positive eigenvalues of (1), (2) are also functionals of p and denoting them by $\lambda_1[p] < \lambda_2[p] < \cdots$, we are able to prove the following result.

THEOREM 2. *The positive eigenvalues of* (1) *with boundary conditions* $u(0) = u(l) = 0$ *and* $\rho(x) > 0$, $x \in [0, l]$ *satisfy the inequality*

$$\lambda_{m+n-1}[p] \geqq (\lambda_{m-1}[p_1^{\alpha}])^{1/\alpha}(\lambda_{n-1}[p_2^{\beta}])^{1/\beta}$$

where $p = p_1 \cdot p_2$, $p_1 > 0$, $p_2 > 0$ *on* $[0, l]$ *and* $1/\alpha + 1/\beta = 1$.

PROOF. We note that u_ν is an eigenfunction of (1), (2) corresponding to the eigenvalue $\lambda_\nu[p]$ if and only if $u_\nu = pu_\nu'$ is an eigenfunction of the reciprocal system

$$\left(\frac{1}{\rho(x)} v'\right)' + \lambda \frac{1}{p(x)} v = 0, \qquad v'(0) = v'(l) = 0 \tag{9}$$

corresponding to the eigenvalue $\lambda_\nu[p]$. We note, however, that the system (9) has zero for an eigenvalue corresponding to the eigenfunction $u_0(x) = \text{const}$. The nonzero eigenvalues of (9) satisfy the following maximum-minimum principle:

Let D denote the space of absolutely continuous functions such that

$$\int_0^l \frac{1}{p} v u_0 dx = \int_0^l \frac{1}{p} v dx = 0.$$

Let V_{n-1} denote the subspace of D spanned by the set $\{v_1, v_2, \cdots, v_{n-1}\}$ taken from D. Denote the Rayleigh quotient corresponding to (9) by

$$R[p, u] = \frac{\int_0^l \frac{1}{\rho} u'^2 dx}{\int_0^l \frac{1}{p} u^2 dx}.$$

Then $\lambda_n[p] = \max_{V_{n-1}} \min_{u \perp V_{n-1}, u \in D} R[p, u]$.

The proof now follows as in the proof of Theorem 1.

Finally, we note that the results presented here are related to those in [3] where generalized means and in particular geometric means are discussed. Also, see reference [4].

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