

# INEQUALITIES FOR THE EIGENVALUES OF POWERS OF FUNCTIONS

DALLAS O. BANKS

In this paper, we give generalizations of a result of A. M. Fink. Specifically, we consider the Sturm-Liouville system

$$\begin{aligned} (1) \quad & (p(x)u')' + \lambda\rho(x)u = 0, \quad x \in (0, l), \\ (2) \quad & p(0)u'(0) - h_0u(0) = 0, \quad p(l)u'(l) + h_1u(l) = 0, \end{aligned}$$

where  $p, \rho$  are integrable functions on  $[0, l]$  with  $p'$  continuous and  $\rho$  positive in a subinterval of  $[0, l]$ . Furthermore,  $h_0, h_1$  are nonnegative numbers with  $h_0, h_1$  approaching  $\infty$  corresponding to the boundary conditions  $u(0) = 0, u(l) = 0$ , respectively. We will be concerned with the dependence of the positive eigenvalues of the system (1), (2) on the function  $\rho$ . We denote them accordingly by  $0 < \lambda_1[\rho] < \lambda_2[\rho] < \dots$ . Fink shows that in the case where (1), (2) reduces to

$$\begin{aligned} u'' + \lambda\rho(x)u &= 0, \quad x \in [0, l], \\ u(0) &= u(l) = 0, \end{aligned}$$

the inequality

$$(3) \quad \lambda_1[\rho^2] \leq (l\lambda_1[\rho]/\pi)^2$$

holds [1]. We prove the following

**THEOREM 1.** *The positive eigenvalues of (1), (2) satisfy the functional inequality*

$$(4) \quad \lambda_{m+n-1}[\rho] \geq (\lambda_m[|\rho_1|^\alpha])^{1/\alpha} (\lambda_n[|\rho_2|^\beta])^{1/\beta}$$

where  $\rho = \rho_1 \cdot \rho_2$  and  $1/\alpha + 1/\beta = 1$ .

When  $\rho_2 \equiv 1$  on  $[0, l]$ ,  $n = m = 1, k = 1, \alpha = \beta = 2$ , (4) reduces to Fink's inequality (3). Furthermore, the proof given here generalizes trivially to higher dimensional problems.

**PROOF OF THEOREM.** We make use of the Rayleigh quotient

$$(5) \quad R[\rho, u] = \frac{\int_0^l u'^2 dx + h_0 u^2(0) + h_1 u^2(l)}{\int_0^l \rho u^2 dx}$$

---

Received by the editors March 28, 1969.

and the max-min characterization of the eigenvalues, i.e.,

$$(6) \quad \lambda_{n+1}[\rho] = \max_{[v_1, \dots, v_n]} d[v_1, \dots, v_n],$$

where  $[v_1, \dots, v_n]$  is the subspace spanned by  $n$  smooth functions  $v_1, \dots, v_n$  and

$$(7) \quad d[v_1, \dots, v_n] = \min R[\rho, u],$$

where the minimum is taken over all smooth functions satisfying the conditions

$$(u, v_k) = \int_0^l \rho u v_k dx = 0$$

( $k=1, \dots, n$ ) and  $\int_0^l \rho u^2 dx \geq 0$  (see [2, p. 214]). By Hölder's inequality, we have

$$(8) \quad \begin{aligned} \int_0^l \rho u^2 dx &= \int_0^l (\rho_1 u^{2/\alpha})(\rho_2 u^{2/\beta}) dx \\ &\leq \left( \int_0^l |\rho_1|^\alpha u^2 dx \right)^{1/\alpha} \left( \int_0^l |\rho_2|^\beta u^2 dx \right)^{1/\beta} \end{aligned}$$

with  $1/\alpha + 1/\beta = 1$ .

Let  $U = [u_1, u_2, \dots, u_{m-1}]$  where  $\{u_k\}_{k=1}^{m-1}$  is the set of eigenfunctions corresponding to  $\{\lambda_k[|\rho_1|]\}_{k=1}^{m-1}$  and let  $V = [v_1, v_2, \dots, v_{n-1}]$  where  $\{v_k\}_{k=1}^{n-1}$  is the set corresponding to  $\{\lambda_k[|\rho_2|]\}_{k=1}^{n-1}$ .

If  $W = [u_1, \dots, u_{m-1}, v_1, \dots, v_{n-1}]$ , then (6) and (7) yield

$$\lambda_{m+n-1}[\rho] \geq \min R[\rho_1 u],$$

where the minimum is taken over the set  $\{u: u \in C', u \perp W, \int_0^l \rho u^2 dx \geq 0\}$ . By (8) and another application of the max-min principle we have

$$\begin{aligned} \lambda_{m+n-1}[\rho] &\geq \min_{u \perp W} (R[|\rho_1|^\alpha, u])^{1/\alpha} (R[|\rho_2|^\beta, u])^{1/\beta} \\ &\geq (\min_{u \perp U} R[|\rho_1|^\alpha, u])^{1/\alpha} (\min_{u \perp V} R[|\rho_2|^\beta, u])^{1/\beta} \\ &= (\lambda_{m-1}[|\rho_1|^\alpha])^{1/\alpha} (\lambda_{n-1}[|\rho_2|^\beta])^{1/\beta}. \end{aligned}$$

Noting that the positive eigenvalues of (1), (2) are also functionals of  $p$  and denoting them by  $\lambda_1[p] < \lambda_2[p] < \dots$ , we are able to prove the following result.

**THEOREM 2.** *The positive eigenvalues of (1) with boundary conditions  $u(0) = u(l) = 0$  and  $\rho(x) > 0, x \in [0, l]$  satisfy the inequality*

$$\lambda_{m+n-1}[p] \cong (\lambda_{m-1}[p_1^\alpha])^{1/\alpha} (\lambda_{n-1}[p_2^\beta])^{1/\beta}$$

where  $p = p_1 \cdot p_2$ ,  $p_1 > 0$ ,  $p_2 > 0$  on  $[0, l]$  and  $1/\alpha + 1/\beta = 1$ .

PROOF. We note that  $u_v$  is an eigenfunction of (1), (2) corresponding to the eigenvalue  $\lambda_v[p]$  if and only if  $u_v = pu'_v$  is an eigenfunction of the reciprocal system

$$(9) \quad \left( \frac{1}{\rho(x)} v' \right)' + \lambda \frac{1}{p(x)} v = 0, \quad v'(0) = v'(l) = 0$$

corresponding to the eigenvalue  $\lambda_v[p]$ . We note, however, that the system (9) has zero for an eigenvalue corresponding to the eigenfunction  $u_0(x) = \text{const}$ . The nonzero eigenvalues of (9) satisfy the following maximum-minimum principle:

Let  $D$  denote the space of absolutely continuous functions such that

$$\int_0^l \frac{1}{p} v u_0 dx = \int_0^l \frac{1}{p} v dx = 0.$$

Let  $V_{n-1}$  denote the subspace of  $D$  spanned by the set  $\{v_1, v_2, \dots, v_{n-1}\}$  taken from  $D$ . Denote the Rayleigh quotient corresponding to (9) by

$$R[p, u] = \frac{\int_0^l \frac{1}{\rho} u'^2 dx}{\int_0^l \frac{1}{p} u^2 dx}.$$

Then  $\lambda_n[p] = \max_{V_{n-1}} \min_{u \perp V_{n-1}, u \in D} R[p, u]$ .

The proof now follows as in the proof of Theorem 1.

Finally, we note that the results presented here are related to those in [3] where generalized means and in particular geometric means are discussed. Also, see reference [4].

#### REFERENCES

1. A. M. Fink, *Eigenvalues of the square of a function*, Proc. Amer. Math. Soc. 20 (1969), 73-74.
2. E. Kamke, *Differentialgleichungen, Lösungsmethoden und Lösungen*, Chelsea, New York, 1948.
3. D. O. Banks, *Generalized means of eigenvalues*, J. Math. Anal. Appl. 23 (1968), 409-420.
4. L. Janoš, *Functional properties of the spectrum of boundary value problems*, Časopis Pěst. Mat. 23 (1960), 461-474.