# A THEOREM OF ALIEV

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1. **Introduction.** We are concerned with the $n$th ($n \geqq 3$) order linear differential equation

$$l_n[y] \equiv y^{(n)} + \sum_{k=0}^{n-1} p_k(x) y^{(k)} = 0$$

where the coefficients are continuous on $(-\infty, \infty)$. The results in this paper generalize the well-known result that the first conjugate point $\eta_1(t)$ for $l_3[y]=0$ satisfies

$$\eta_1(t) = \min[r_{21}(t), r_{12}(t)]$$

(see Definition 2). Aliev [**1**] proved for $l_4[y]=0$ that

$$r_{1111}(t) = \min[r_{121}(t), r_{112}(t)]$$

and purported to prove that

$$r_{1111}(t) = \min[r_{211}(t), r_{112}(t)] \tag{1}$$

but his proof is incorrect. Since $\eta_1(t) = r_{1111}(t)$ [**2**], [**3**], we have $\eta_1(t) = \min[r_{121}(t), r_{112}(t)]$. Theorem 1 gives a much easier proof of this result, establishes the validity of (1) (this was left as an open question in [**4**]), and gives an $n$th order generalization of these results. The simplicity of the Theorem 1 is due to a theorem of Sherman [**5**] which gives that if $b > \eta_1(t)$, then there is a nontrivial solution of $l_n[y]=0$ with a simple zero at $t$ and whose first $n$ zeros on $[t, b)$ are simple zeros.

2. **Definitions and main result.** Before we state the main result we make the following definitions.

DEFINITION 1. A nontrivial solution $y$ of $l_n[y]=0$ is said to have a $i_1 - i_2 - \cdots - i_\nu$ ($\nu = 2, \cdots, n$, $\sum_{k=1}^{\nu} i_k = n$, $1 \leqq i_\nu \leqq n-1$) distribution of zeros on $[t, b]$ provided there are numbers $t_1, \cdots, t_\nu$ such that $t \leqq t_0 < t_1 < \cdots < t_\nu \leqq b$ and $y$ has a zero at each $t_k$ of order at least $i_k$.

DEFINITION 2. The extended real number $r_{i_1 i_2 \cdots i_\nu}(t)$ is the infimum of the set of $b > t$ such that there is a nontrivial solution $y$ of $l_n[y]=0$ having an $i_1 - i_2 - \cdots - i_\nu$ distribution of zeros on $[t, b]$.

REMARK 1. If $t \leqq t_1 < t_2 < \cdots < t_\nu < r_{i_1 i_2 \cdots i_\nu}(t) \leqq \infty$, then there is a unique solution $u(x)$ of $l_n[y]=0$ satisfying

$$y^{(j)}(t_k) = A_{jk},$$

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$k=1, 2, \cdots, \nu; j=0, 1, \cdots, i_k$ where the $A_{jk}$ are constants.

DEFINITION 3. For $1 \leqq p \leqq n-1$

$$s_p(t) = r_{i_1 i_2 \cdots i_{n-1}}(t)$$

where $i_p = 2$ and $i_k = 1$ for $k \neq p$.

We now state our main result.

THEOREM 1. *For* $1 \leqq j \neq k \leqq n-1$

$$\eta_1(t) = \min[s_j(t), s_k(t)].$$

PROOF. If $\eta_1(t) = \infty$, then the theorem is obvious and, hence, we can assume that $\eta_1(t)$ exists. Clearly $\eta_1(t) \leqq \rho(t) \equiv \min[s_j(t), s_k(t)]$, and so it suffices to show that the assumption $\eta_1(t) < \rho(t)$ leads to a contradiction. By Theorem **1** in [**5**] there is a nontrivial solution $u(x)$ of $l_n[y]=0$ whose first $n$ zeros, say $x_k$, $1 \leqq k \leqq n$, are simple zeros where $t = x_1 < x_2 < \cdots < x_n < \rho(t)$. We can assume that $j < k$. Since $t = x_1 < x_2 < \cdots < x_n < s_j(t)$ there is a unique solution $v(x)$ of $l_n[y]=0$ satisfying

$$y(x_i) = 0, \qquad y'(x_j) = 0, \qquad y(x_k) = 1,$$

where $i = 1, \cdots, k-1, k+2, \cdots, n$ unless $k = n-1$ in which case $i = 1, 2, \cdots, n-2$ (see Remark 1). It is easy to see that $v(x) > 0$ for $x_k \leqq x \leqq x_{k+1}$. It follows from Lemma **1.1** in [**6**] that there is a nontrivial linear combination of $u(x)$ and $v(x)$ with a double zero in $(x_k, x_{k+1})$. But this same linear combination has zeros at $x_i$, $i = 1, \cdots, k-1, k+2, \cdots, n$ unless $k = n-1$ in which case $i = 1, 2, \cdots, n-2$. This contradicts $t = x_1 < x_2 < \cdots < x_n < s_k(t)$ and the theorem is proved.

It follows from Theorem 1 that at most one of the numbers $s_p(t)$, $1 \leqq p \leqq n-1$, is greater than $\eta_1(t)$. Many examples can be given to show that we do not have $\eta_1(t) = s_p(t)$ for $1 \leqq p \leqq n-1$. For $n=3$ see Hanan [**7**]. A simple example for $n=4$ is $y^{iv} + y' = 0$ for which we have $\eta_1(t) = s_1(t) = s_2(t) \approx t + 5.9 < s_3(t) = \infty$ [**8**]. For those equations of the form

$$y^{iv} + p(x)y = 0, \qquad p(x) < 0$$

for which $\eta_1(t)$ exists we have $\eta_1(t) = s_1(t) = s_3(t) < s_2(t) = \infty$ [**4**], [**6**].

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