A THEOREM OF ALIEV

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1. **Introduction.** We are concerned with the nth ($n \geqq 3$) order linear differential equation

$$l_n[y] \equiv y^{(n)} + \sum_{k=0}^{n-1} p_k(x) y^{(k)} = 0$$

where the coefficients are continuous on $(-\infty, \infty)$. The results in this paper generalize the well-known result that the first conjugate point $\eta_1(t)$ for $l_3[y]=0$ satisfies

$$\eta_1(t) = \min[r_{21}(t), r_{12}(t)]$$

(see Definition 2). Aliev [1] proved for $l_4[y]=0$ that

$$r_{1111}(t) = \min[r_{121}(t), r_{112}(t)]$$

and purported to prove that

$$(1) \qquad r_{1111}(t) = \min[r_{211}(t), r_{112}(t)]$$

but his proof is incorrect. Since $\eta_1(t) = r_{1111}(t)$ [2], [3], we have $\eta_1(t) = \min[r_{121}(t), r_{112}(t)]$. Theorem 1 gives a much easier proof of this result, establishes the validity of (1) (this was left as an open question in [4]), and gives an nth order generalization of these results. The simplicity of the Theorem 1 is due to a theorem of Sherman [5] which gives that if $b > \eta_1(t)$, then there is a nontrivial solution of $l_n[y]=0$ with a simple zero at t and whose first n zeros on $[t, b)$ are simple zeros.

2. **Definitions and main result.** Before we state the main result we make the following definitions.

DEFINITION 1. A nontrivial solution y of $l_n[y]=0$ is said to have a $i_1 - i_2 - \cdots - i_\nu$ ($\nu = 2, \cdots, n$, $\sum_{k=1}^{\nu} i_k = n$, $1 \leqq i_\nu \leqq n-1$) distribution of zeros on $[t, b]$ provided there are numbers $t_1, \cdots, t_\nu$ such that $t \leqq t_0 < t_1 < \cdots < t_\nu \leqq b$ and y has a zero at each t_k of order at least i_k.

DEFINITION 2. The extended real number $r_{i_1 i_2 \cdots i_\nu}(t)$ is the infimum of the set of $b > t$ such that there is a nontrivial solution y of $l_n[y]=0$ having an $i_1 - i_2 - \cdots - i_\nu$ distribution of zeros on $[t, b]$.

REMARK 1. If $t \leqq t_1 < t_2 < \cdots < t_\nu < r_{i_1 i_2 \cdots i_\nu}(t) \leqq \infty$, then there is a unique solution $u(x)$ of $l_n[y]=0$ satisfying

$$y^{(j)}(t_k) = A_{jk},$$

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$k=1, 2, \cdots, \nu; j=0, 1, \cdots, i_k$ where the A_{jk} are constants.

DEFINITION 3. For $1 \leqq p \leqq n-1$

$$s_p(t) = r_{i_1 i_2 \cdots i_{n-1}}(t)$$

where $i_p=2$ and $i_k=1$ for $k \neq p$.

We now state our main result.

THEOREM 1. *For* $1 \leqq j \neq k \leqq n-1$

$$\eta_1(t) = \min[s_j(t), s_k(t)].$$

PROOF. If $\eta_1(t) = \infty$, then the theorem is obvious and, hence, we can assume that $\eta_1(t)$ exists. Clearly $\eta_1(t) \leqq \rho(t) \equiv \min[s_j(t), s_k(t)]$, and so it suffices to show that the assumption $\eta_1(t) < \rho(t)$ leads to a contradiction. By Theorem 1 in [5] there is a nontrivial solution $u(x)$ of $l_n[y]=0$ whose first n zeros, say x_k, $1 \leqq k \leqq n$, are simple zeros where $t=x_1<x_2 < \cdots < x_n < \rho(t)$. We can assume that $j<k$. Since $t=x_1<x_2< \cdots < x_n < s_j(t)$ there is a unique solution $v(x)$ of $l_n[y]=0$ satisfying

$$y(x_i) = 0, \qquad y'(x_j) = 0, \qquad y(x_k) = 1,$$

where $i=1, \cdots, k-1, k+2, \cdots, n$ unless $k=n-1$ in which case $i=1, 2, \cdots, n-2$ (see Remark 1). It is easy to see that $v(x)>0$ for $x_k \leqq x \leqq x_{k+1}$. It follows from Lemma 1.1 in [6] that there is a nontrivial linear combination of $u(x)$ and $v(x)$ with a double zero in (x_k, x_{k+1}). But this same linear combination has zeros at x_i, $i=1, \cdots, k-1, k+2, \cdots, n$ unless $k=n-1$ in which case $i=1, 2, \cdots, n-2$. This contradicts $t=x_1<x_2< \cdots <x_n<s_k(t)$ and the theorem is proved.

It follows from Theorem 1 that at most one of the numbers $s_p(t)$, $1 \leqq p \leqq n-1$, is greater than $\eta_1(t)$. Many examples can be given to show that we do not have $\eta_1(t)=s_p(t)$ for $1 \leqq p \leqq n-1$. For $n=3$ see Hanan [7]. A simple example for $n=4$ is $y^{iv}+y'=0$ for which we have $\eta_1(t)=s_1(t)=s_2(t) \approx t+5.9 < s_3(t) = \infty$ [8]. For those equations of the form

$$y^{iv} + p(x)y = 0, \qquad p(x) < 0$$

for which $\eta_1(t)$ exists we have $\eta_1(t)=s_1(t)=s_3(t)<s_2(t) = \infty$ [4], [6].

REFERENCES

1. R. G. Aliev, *On certain properties of solutions of ordinary differential equations of fourth order*, Sb. Aspirantsh, Kazan Institute (1964), 15–30. (Russian).

2. P. Hartman, *Unrestricted n-parameter families*, Rend. Circ. Mat. Palermo (2) **7** (1958), 123–142.

3. Z. Opial, *On a theorem of O. Arama*, J. Differential Equations **3** (1967), 88–91.

4. A. C. Peterson, *Distribution of zeros of solutions of a fourth order differential equation*, Pacific J. Math. (to appear).

5. T. L. Sherman, *Conjugate points and simple zeros for ordinary linear differential equations*, Trans. Amer. Math. Soc. (to appear).

6. W. Leighton and Z. Nehari, *On the oscillation of solutions of self-adjoint linear differential equations of the fourth order*, Trans. Amer. Math. Soc. **89** (1958), 325–377.

7. M. Hanan, *Oscillation criteria for third-order linear differential equations*, Pacific J. Math. **11** (1961), 919–944.

8. R. G. Aliev, *On the sign of the Green's function for a boundary-value problem for a fourth order differential equation*, Izv. Vysč, Učebn. Zaved. Matematika **1964**, no. 6 (43), 3–9. (Russian)

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