A THEOREM OF ALIEV
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1. Introduction. We are concerned with the \( n \)th \((n \geq 3)\) order linear differential equation

\[
 l_n[y] = y^{(n)} + \sum_{k=0}^{n-1} p_k(x)y^{(k)} = 0
\]

where the coefficients are continuous on \((-\infty, \infty)\). The results in this paper generalize the well-known result that the first conjugate point \( \eta_1(t) \) for \( l_3[y] = 0 \) satisfies

\[
 \eta_1(t) = \min[r_{21}(t), r_{12}(t)]
\]

(see Definition 2). Aliev [1] proved for \( l_4[y] = 0 \) that

\[
 r_{1111}(t) = \min[r_{121}(t), r_{112}(t)]
\]

and purported to prove that

\[
 (1) \quad r_{1111}(t) = \min[r_{211}(t), r_{112}(t)]
\]

but his proof is incorrect. Since \( \eta_1(t) = r_{1111}(t) \) [2], [3], we have \( \eta_1(t) = \min[r_{211}(t), r_{112}(t)] \). Theorem 1 gives a much easier proof of this result, establishes the validity of (1) (this was left as an open question in [4]), and gives an \( n \)th order generalization of these results. The simplicity of the Theorem 1 is due to a theorem of Sherman [5] which gives that if \( b > \eta_1(t) \), then there is a nontrivial solution of \( l_n[y] = 0 \) with a simple zero at \( t \) and whose first \( n \) zeros on \([t, b)\) are simple zeros.

2. Definitions and main result. Before we state the main result we make the following definitions.

Definition 1. A nontrivial solution \( y \) of \( l_n[y] = 0 \) is said to have a \( i_1 - i_2 - \cdots - i_v \) \((v = 2, \cdots, n, \sum_{k=1}^v i_k = n, 1 \leq i_v \leq n - 1)\) distribution of zeros on \([t, b)\) provided there are numbers \( t_1, \cdots, t_v \) such that \( t_0 < t_1 < \cdots < t_v \leq b \) and \( y \) has a zero at each \( t_k \) of order at least \( i_k \).

Definition 2. The extended real number \( r_{i_1i_2\cdots i_v}(t) \) is the infimum of the set of \( b > t \) such that there is a nontrivial solution \( y \) of \( l_n[y] = 0 \) having an \( i_1 - i_2 - \cdots - i_v \) distribution of zeros on \([t, b)\).

Remark 1. If \( t \leq t_1 < t_2 < \cdots < t_v < t_{i_1i_2\cdots i_v}(t) \leq \infty \), then there is a unique solution \( u(x) \) of \( l_n[y] = 0 \) satisfying

\[
 y^{(j)}(t_k) = A_{jk},
\]
$k=1, 2, \ldots, v$; $j=0, 1, \ldots, i_k$ where the $A_{jk}$ are constants.

**Definition 3.** For $1 \leq p \leq n-1$

$$s_p(t) = r_{i_1i_2\cdots i_{n-1}}(t)$$

where $i_p = 2$ and $i_k = 1$ for $k \neq p$.

We now state our main result.

**Theorem 1.** For $1 \leq j \neq k \leq n-1$

$$\eta_1(t) = \min[s_j(t), s_k(t)].$$

**Proof.** If $\eta_1(t) = \infty$, then the theorem is obvious and, hence, we can assume that $\eta_1(t)$ exists. Clearly $\eta_1(t) \leq \rho(t) = \min[s_j(t), s_k(t)]$, and so it suffices to show that the assumption $\eta_1(t) < \rho(t)$ leads to a contradiction. By Theorem 1 in [5] there is a nontrivial solution $u(x)$ of $l_n[y] = 0$ whose first $n$ zeros, say $x_k$, $1 \leq k \leq n$, are simple zeros where $t = x_1 < x_2 < \cdots < x_n < \rho(t)$. We can assume that $j \neq k$. Since $t = x_1 < x_2 < \cdots < x_n < s_j(t)$ there is a unique solution $v(x)$ of $l_n[y] = 0$ satisfying

$$y(x_i) = 0, \quad y'(x_j) = 0, \quad y(x_k) = 1,$$

where $i = 1, \ldots, k-1, k+2, \ldots, n$ unless $k = n-1$ in which case $i = 1, 2, \ldots, n-2$ (see Remark 1). It is easy to see that $v(x) > 0$ for $x_k \leq x \leq x_{k+1}$. It follows from Lemma 1.1 in [6] that there is a nontrivial linear combination of $u(x)$ and $v(x)$ with a double zero in $(x_k, x_{k+1})$. But this same linear combination has zeros at $x_i$, $i = 1, \ldots, k-1, k+2, \ldots, n$ unless $k = n-1$ in which case $i = 1, 2, \ldots, n-2$. This contradicts $t = x_1 < x_2 < \cdots < x_n < s_k(t)$ and the theorem is proved.

It follows from Theorem 1 that at most one of the numbers $s_p(t)$, $1 \leq p \leq n-1$, is greater than $\eta_1(t)$. Many examples can be given to show that we do not have $\eta_1(t) = s_p(t)$ for $1 \leq p \leq n-1$. For $n = 3$ see Hanan [7]. A simple example for $n = 4$ is $y'' + y' = 0$ for which we have $\eta_1(t) = s_1(t) = s_3(t) = t + 5.9 < s_2(t) = \infty$ [8]. For those equations of the form

$$y''' + \rho(x)y = 0, \quad \rho(x) < 0$$

for which $\eta_1(t)$ exists we have $\eta_1(t) = s_1(t) = s_3(t) < s_2(t) = \infty$ [4], [6].

**References**


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