AN ASYMPTOTIC PROPERTY OF SOLUTIONS OF WAVE EQUATIONS

JEROME A. GOLDSTEIN

Let X be a complex Hilbert space with inner product $\langle\cdot, \cdot\rangle$ and norm $\|\cdot\|$. Let A be a selfadjoint (in general unbounded) linear operator on X satisfying

$$\langle Ax, x\rangle \geqq 0 \qquad \text{for all } x \in D(A), \tag{1}$$

where $D(A)$ denotes the domain of A. We shall consider "wave" equations of the form

$$u''(t) + Au(t) = 0 \qquad (t \in \boldsymbol{R}) \tag{2}$$

$({}' = d/dt)$ with initial data

$$u(0) = f_1 \in D(A), \qquad u'(0) = f_2 \in D(A^{1/2}). \tag{3}$$

THEOREM. (i) *Let A be a selfadjoint operator on X satisfying* (1). *Then for any $f_1 \in D(A)$ and for any $f_2 \in D(A^{1/2})$, the initial value problem* (2), (3) *has a unique twice strongly continuously differentiable solution. Let*

$$\Pi_f = \|A^{1/2}f_1\|^2 + \|f_2\|^2.$$

Then

$$\|A^{1/2}u(t)\|^2 + \|u'(t)\|^2 = \Pi_f \tag{4}$$

for all $t \in \boldsymbol{R}$. Let $\{E_\lambda, \lambda \in \boldsymbol{R}\}$ be the resolution of the identity associated with $A^{1/2}$. Then

$$\lim_{|t| \to \infty} \|A^{1/2}u(t)\|^2 = \lim_{|t| \to \infty} \|u'(t)\|^2 = \Pi_f/2 \tag{5}$$

for all choices of initial data as in (3) *if and only if*

$$\lim_{|t| \to \infty} \int_0^\infty e^{it\lambda} d_\lambda(\|E_\lambda x\|^2) = 0 \qquad \text{for all } x \in X. \tag{6}$$

(ii) *If, in addition,* 0 *belongs to the resolvent set of A, let*

$$\Gamma_f = \|f_1\|^2 + \|A^{-1/2}f_2\|^2.$$

Then

$$\|u(t)\|^2 + \|A^{-1/2}u'(t)\|^2 = \Gamma_f \tag{7}$$

Received by the editors February 11, 1969.

for all $t \in \boldsymbol{R}$. *Moreover,*

$$(8) \qquad \lim_{|t| \to \infty} \|u(t)\|^2 = \lim_{|t| \to \infty} \|A^{-1/2}u'(t)\|^2 = \Gamma_f/2$$

if and only if (6) *holds.*

(iii) *In order that* (6) *holds, it is necessary that the function* $\lambda \to E_\lambda x$ *be strongly continuous for each* $x \in X$, *and it is sufficient that* $\lambda \to E_\lambda x$ *be strongly absolutely continuous for each* $x \in X$.

Let $\{F_\lambda, \lambda \in \boldsymbol{R}\}$ be the resolution of the identity associated with A. Then $E_\lambda = F_{\lambda^2}$ for all $\lambda \in \boldsymbol{R}$. Thus for $x \in X$, $\lambda \to F_\lambda x$ is strongly continuous or strongly absolutely continuous if and only if $\lambda \to E_\lambda x$ is.

The above theorem is a generalization of a recent result of M. Shinbrot [**3**]. Shinbrot proved that the strong absolute continuity of $\lambda \to F_\lambda x$ for each $x \in X$ was a sufficient condition for (5) and (8) to hold, and that the strong continuity of $\lambda \to F_\lambda x$ was a necessary condition. Condition (6) is thus the condition, weaker than strong absolute continuity of $\lambda \to F_\lambda$ and stronger than strong continuity of $\lambda \to F_\lambda$, which is both necessary and sufficient for (5) and (8) to hold.

Our proof is by semigroup methods in contrast to Shinbrot's use of spectral theoretic methods.

PROOF OF THE THEOREM. Assume first that 0 is in the resolvent set of A. The initial value problem (2), (3) can be written as

$$U'(t) = MU(t), \qquad U(0) = f$$

where

$$U(t) = \begin{pmatrix} u(t) \\ u'(t) \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

The above equation will be viewed as a differential equation in the Hilbert space $Y = D(A^{1/2}) \times X$, equipped with the (energy) norm

$$\left| \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right| = (\|A^{1/2}y_1\|^2 + \|y_2\|^2)^{1/2}.$$

Then, by Lemma 3.1 of [**1**], M, with domain $D(A) \times D(A^{1/2})$, is the infinitesimal generator of a strongly continuous group $\{T_t, t \in \boldsymbol{R}\}$ of unitary operators on Y. T_t is given by the formula

$$(9) \qquad T_t \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cosh(tB)y_1 + \sinh(tB)B^{-1}y_2 \\ \cosh(tB)y_2 + \sinh(tB)By_1 \end{pmatrix}$$

where $B = iA^{1/2}$, and $\cosh(tB)$, $\sinh(tB)$ are defined in the obvious manner in terms of the unitary group $\{\exp(sB), s \in \boldsymbol{R}\}$ generated by B. The first component of $U(t)=T_t f$ is thus the unique twice strongly continuously differentiable solution of (2), (3). The unitary property of T_t proves (4), and (7) follows from

$$\|u(t)\|^2 + \|B^{-1}u'(t)\|^2 = \left|\begin{pmatrix} B^{-1}u(t) \\ B^{-1}u'(t) \end{pmatrix}\right|^2 = \left|\begin{pmatrix} B^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix} T_t f\right|^2$$
$$= \left|T_t \begin{pmatrix} B^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix} f\right|^2 = \left|\begin{pmatrix} B^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix} f\right|^2 = \Gamma_f$$

for all $t \in \boldsymbol{R}$.

Note that if V is any unitary operator on X and if $x, y \in X$, then

$$\|x + Vy\|^2 = \|x\|^2 + \|y\|^2 + 2 \operatorname{Re} \langle x, Vy\rangle. \tag{10}$$

Now let $V = \exp(-2tB)$. Then by (9), the definition of the norm in Y, and the fact that $\exp(tB)$ is unitary,

$$\begin{aligned} 4\|u'(t)\|^2 &= \|\exp(tB)\{(I + V)f_2 + (I - V)Bf_1\}\|^2 \\ &= \|f_2 + Bf_1 + V(f_2 - Bf_1)\|^2 \\ &= 2(\|f_2\|^2 + \|Bf_1\|^2) + 2 \operatorname{Re} \langle f_2 + Bf_1, V(f_2 - Bf_1)\rangle \end{aligned}$$

by (10) and the parallelogram law. Thus

$$2\|u'(t)\|^2 = \Pi_f + \operatorname{Re} \langle f_2 + Bf_1, V(f_2 - Bf_1)\rangle. \tag{11}$$

A similar calculation yields

$$2\|u(t)\|^2 = \Gamma_f + \operatorname{Re}\langle f_1 + B^{-1}f_2, V(f_1 - B^{-1}f_2)\rangle. \tag{12}$$

Hence by (11) and (4), (5) holds if and only if

$$\lim_{|t|\to\infty} \operatorname{Re}\langle f_2 + Bf_1, V(f_2 - Bf_1)\rangle = 0 \tag{13}$$

for all $f_1 \in D(A)$ (i.e. for all $Bf_1 \in D(B)$) and for all $f_2 \in D(B)$. Noting that $V = \exp(-2tB)$, (13) is thus equivalent to

$$\lim_{|t|\to\infty} \langle \exp(tB)x, y\rangle = 0$$

for all $x, y \in D(B)$ and hence for all $x, y \in \mathrm{Cl}(D(B)) = X$. Polarization shows that this is equivalent to

$$\lim_{|t|\to\infty} \langle \exp(tB)x, x\rangle = 0 \qquad \text{for all } x \in X. \tag{14}$$

Therefore (11), (12) and the above calculation show that (5), (14) and (8) are equivalent. If $\{E_\lambda, \lambda \in \boldsymbol{R}\}$ is the resolution of the identity associated with $A^{1/2}$, then since $B = iA^{1/2}$,

$$\langle \exp(tB)x,\, x\rangle = \int_0^\infty e^{it\lambda} d_\lambda \langle E_\lambda x,\, x\rangle,$$

so that (14) is equivalent to (6). Thus (i) and (ii) are proved in the case that 0 belongs to the resolvent set of A.

For the general case in (i), let A be a selfadjoint operator on X satisfying (1). Let $A_n = A + 2n^{-1}A^{1/2} + n^{-2}I$, so that $A_n^{1/2} = A^{1/2} + n^{-1}I$. Let $B_n = iA_n^{1/2}$. Then if u_n is the unique strong solution of

$$u_n''(t) + A_n u_n(t) = 0 \qquad (t \in \boldsymbol{R})$$

with initial data (3), then

$$u_n'(t) = \cosh(tB_n)f_2 + \sinh(tB_n)B_n f_1, \tag{15}$$

as was shown previously. Since

$$\exp(tB_n) = e^{it/n} \exp(tB)$$

where $B = iA^{1/2}$, it follows that as $n \to \infty$, u_n' converges uniformly on compact intervals to a necessarily strongly continuous function v given by

$$v(t) = \cosh(tB)f_2 + \sinh(tB)Bf_1.$$

Let $u(t) = \int_0^t v(s)ds + f_1$. Then

$$u_n(t) = \int_0^t u_n'(s)ds + f_1 \to u(t)$$

as $n \to \infty$ uniformly on compact intervals. Hence

$$u''(t) = v'(t) = \sinh(tB)Bf_2 + \cosh(tB)B^2 f_1$$

since $f_1 \in D(B)$, $f_2 \in D(B^2)$ by (3). Therefore u is twice strongly continuously differentiable and

$$\begin{aligned} u''(t) + Au(t) &= \sinh(tB)Bf_2 + \cosh(tB)B^2 f_1 \\ &\quad - B\int_0^t \{\cosh(tB)Bf_2 + \sinh(sB)B^2 f_1\}\, ds = 0 \end{aligned}$$

since $\int_0^t \exp(sB)x\; ds \in D(B)$ for all $x \in X$ and $B\int_0^t \exp(sB)x\; ds = \exp(tB)x - x$. Moreover, (4) holds. Thus u is the unique solution of the initial value problem (2), (3), the uniqueness being a consequence of (4) and linearity.

Let $V = \exp(-2tB)$ as before. Then from (15) it follows that (11) holds, by the same proof as before. Thus (5) holds if and only if (13) holds. Since our proof of the equivalence of (13), (14), and (6) did not require that A be invertible, we conclude that (5) and (6) are equivalent.

We remark that it is tempting to try to deduce the general case of (i) from the special case that A is invertible by applying the Trotter-Kato semigroup convergence theorem [4], [2], [5] to the groups

$$T_t^{(n)} = \left\{\exp\left(t\begin{pmatrix} 0 & I \\ -A_n & 0\end{pmatrix}\right),\ t \in \boldsymbol{R}\right\},$$

where $A_n^{1/2} = A^{1/2} + n^{-1}I$. However, the hypotheses of the Trotter-Kato theorem are not satisfied in this case since $\sup_{n,t} |T_t^{(n)}| = \infty$, where $|\cdot|$ is any norm which makes $Y = D(A^{1/2}) \times X$ into a Banach space.

It remains to prove (iii). If $\lambda \rightarrow E_\lambda x$ is not strongly continuous on $\boldsymbol{R}$ for each $x \in X$, then $A^{1/2}$ has an eigenvalue λ. Let x be a corresponding (nonzero) eigenvector. Then $\exp(itA^{1/2})x = e^{it\lambda}x$, and so

$$\langle \exp(itA^{1/2})x, x\rangle = e^{it\lambda}\|x\|^2,$$

which does not converge to 0 as $|t| \rightarrow \infty$; hence (14) fails to hold and so (6) fails to hold.

Finally, let $g_x(\lambda) = \|E_\lambda x\|^2$. Suppose that g_x is absolutely continuous for each $x \in X$. g_x is also monotone nondecreasing and bounded. Hence its derivative, g_x', belongs to $L^1(0, \infty)$. Therefore for each $x \in X$,

$$\int_0^\infty e^{it\lambda} d_\lambda(\|E_\lambda x\|^2) = \int_0^\infty e^{it\lambda} g_x'(\lambda) d\lambda \rightarrow 0$$

as $|t| \rightarrow \infty$ by the Riemann-Lebesgue lemma, and so (6) holds. This completes the proof.

References

1. J. A. Goldstein, *Time dependent hyperbolic equations*, J. Functional Analysis **3**(1969).

2. T. Kato, *Remarks on pseudo-resolvents and infinitesimal generators of semi-groups*, Proc. Japan Acad. **35** (1959), 467–468.

3. M. Shinbrot, *Asymptotic behavior of solutions of abstract wave equations*, Proc. Amer. Math. Soc. **19** (1968), 1403–1406.

4. H. F. Trotter, *Approximation of semi-groups of operators*, Pacific J. Math. **8** (1958), 887–919.

5. K. Yosida, *Functional analysis*, Springer-Verlag, Berlin, 1965.

Tulane University