TAMENESS IMPLIED BY EXTENDING A HOMEOMORPHISM TO A POINT

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A 2-sphere is a homeomorphic image of the unit sphere B defined in E^3 by the equation $x^2+y^2+z^2=1$. A 2-sphere S is defined to be *tame* in E^3 if there exists a homeomorphism of S onto B that can be extended to a homeomorphism of E^3 onto itself. Stated another way, S is tame if some homeomorphism of E^3-S onto E^3-B exists that can be extended to take S onto B. The main result in this note states that S is tame if, for each point p of S, a homeomorphism h of E^3-S onto E^3-B exists such that h extends to take p onto some point $b\in B$. Actually such a homeomorphism need exist only for each point of a subset of S whose complement in S is a closed 0-dimensional set.

The above condition can also be used to characterize "one sided" tameness; that is, a crumpled cube C is a 3-cell if and only if for each point p of Bd C there exists a point $b\in B$ and a homeomorphism h of Int $C\cup p$ onto Int $B\cup b$. In this case such a homeomorphism need exist only for the piercing points of C. Perhaps of interest in this respect are those crumpled cubes C in E^3 having the property that Bd C is locally tame from Int C modulo a 0-dimensional set W of points where Bd C is wild from Int C. Such an example has been described by Fox and Artin [**7**]. In case W consists of a point p it can be deduced from a result by Doyle and Hocking [**6**] that no homeomorphism exists between Int $C\cup q$ and Int $B\cup b$ as long as $q\neq p$ $(b\in B)$. It follows from Theorem 2 that no such homeomorphism exists for any point of Bd C.

We also show that S is tame if for each p in S and for each $\epsilon>0$ there exist an ϵ-disk D on S containing p and a homeomorphism h of E^3-S onto E^3-B such that h extends to take the boundary of D homeomorphically into B. For example this means that S is tame if it has a curvilinear triangulation T of arbitrarily small mesh such that some homeomorphism of E^3-S onto E^3-B can be extended to take the 1-skeleton of T homeomorphically into B.

Doyle and Hocking [**6**] have defined a $K-R$ 3-manifold M to be a 3-manifold-with-boundary such that Int M and Bd M are topologically E^3 and E^2, respectively. Suppose that M^2 is a closed topo-

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logical plane in a 3-manifold M^3 and U is a component of M^3-M^2. It follows from Corollary 2 that $M^2\cup U$ is a $K-R$ manifold if, for each point $p\in M^2$, $U\cup p$ is topologically Int $B\cup b$ $(b\in B)$. Furthermore, applying [6] we see that the $K-R$ manifold $M^2\cup U$ is topologically $E^2\times[0, 1)$.

In the following definitions and in Lemma 1, V denotes a component of E^3-S and p denotes a point of S. A point set V is called an *open* 3-*cell* if V is homeomorphic to Int B. An ϵ-*set* is a set having diameter less than the positive number ϵ. A 2-sphere S is said to be *locally* α-*capped in* V *at* p if for each $\epsilon>0$ there exists an ϵ-disk R on S such that $p\in$ Int R and for each $\alpha>0$ there is an open ϵ-disk D in $V\cap N(S, \alpha)$ such that Bd D lies in $S-R$ and R lies on the boundary of an ϵ-component of $V-D$. The following lemma is easily proved and appears as Theorem 1 of [10].

LEMMA 1. *If S is locally α-capped in V at p, then V is locally simply connected at p.*

THEOREM 1. *If p is a point of S and there exists a homeomorphism of* Int $S\cup p$ *onto* Int $B\cup b$ $(b\in B)$, *then* Int S *is locally simply connected at p.*

PROOF. We shall show that S is locally α-capped in Int S at p and then rely on Lemma 1 to complete the proof. Let $\epsilon>0$, let U be an $\epsilon/2$-neighborhood of p restricted to Int $S\cup p$, and let h be a homeomorphism as given in the hypothesis. There is a disk E in B such that $h(p)\in\text{Int } E\subset E\subset\text{cl}(h(U))$ and $E\cap\text{cl}(\text{Int } B-h(U))=\emptyset$ (otherwise some sequence $\{h(p_i)\}$ of points in Int $B-h(U)$ converges to $h(p)$ without $\{p_i\}$ converging to p). Let I be the unit interval $[0, 1]$. Since $B\cup$Int B is a 3-cell, there exists a homeomorphism f of $E\times I$ into $B\cup$Int B such that $f(E\times 0)=E$ and $f(E\times(0, 1))\subset h(U)$. Let A' be the open annulus $f(\text{Bd } E\times(0, 1))$ and let $A=h^{-1}(A')$. Then $A\subset U$ and one component M of Bd A lies in $S-p$. Select a disk R such that $p\in\text{Int } R\subset R\subset S-M$, and let $\alpha>0$. If H is the set of all points of Int S within α of S, then $h(H)$ contains $f(E\times(0, t))$ for some t such that $0<t<1$. Let D' be the open disk $f(\text{Bd } E\times(0, t))\cup f(E\times t)$. Then $D=h^{-1}(D')$ is an open ϵ-disk in $N(S, \alpha)$ such that Bd $D=M\subset S-R$, Certainly R lies on the boundary of an ϵ-component of Int $S-D$, so S is locally α-capped in Int S at p.

COROLLARY 1. *Suppose M is a compact connected* 2-*manifold that separates a connected* 3-*manifold M^3, U is a component of M^3-M, and K is a polyhedral homeomorphic image of M in M^3. If there is a component V of M^3-K such that, for each $p\in M$, a point $b\in K$ and a homeomorphism taking $U\cup p$ onto $V\cup b$ exist, then M is tame from U.*

PROOF. The proof given for Theorem 1 is not only valid for 2-spheres but will also hold for 2-manifolds as in the hypothesis. Thus U is locally simply connected at each point of M, and it follows from [2] that M is tame from U.

REMARK. In case M is a 2-sphere Corollary 1 can be improved somewhat by weakening the hypothesis. We proceed toward an improved version, Theorem 3 to follow, by establishing Lemma 2 and Theorem 2 below. Corollary 2 seems of interest relative to $K-R$ manifolds as defined in [6].

COROLLARY 2. *Suppose M is a closed topological plane in a 3-manifold M^3 and U is a component of M^3-M. If, for each $p\in M$, a homeomorphism exists taking $U\cup p$ onto* Int $B\cup b$ $(b\in B)$, *then $M\cup U$ is homeomorphic to $E^2\times[0, 1)$.*

PROOF. As shown in the proof of Theorem 1 U is locally simply connected at each $p\in M$. Thus $M\cup U$ is a 3-manifold with boundary [2]. Now it follows from the corollary in [6] that the $K-R$ manifold $M\cup U$ is topologically $E^2\times[0, 1)$.

LEMMA 2. *Suppose a 2-sphere S in E^3 is locally tame from* Int S *modulo a 0-dimensional set W and some point q in $S-W$ has the following property: There exists an arc A and a sequence $\{S_i\}$ of 2-spheres converging to S such that, for each i, $S_i\subset$* Int S, *$S_i\cap A$ is a point at which A pierces S_i, q is an endpoint of A, and $S\cap A=\{q\}$. Then S is tame from* Int S.

PROOF. Since Int S is an open 3-cell, S is locally tame from Int S modulo one point p [5]. We shall show that S can be locally spanned from Int S at p and then apply the results of [4] to conclude that S is tame from Int S. There is some resemblance between our proof and the proof given for Theorem 1 in [5].

Let $\epsilon>0$ and choose δ such that $0<7\delta<\epsilon$ and $\delta<d(p, A)$. There exists a disk R, a locally polyhedral annulus B, and a simple closed curve J such that $p\in\text{Int } R\subset S$, $\text{Bd } R\subset B$, $B-\text{Bd } R\subset\text{Int } S$, $\text{Diam } (B\cup R)<\delta$, $J\cap S=\{p, q\}$, $J\cap B=\emptyset$, $A\subset J$, and J pierces S at both p and q. Let $J_1=\text{Bd } B-\text{Bd } R$. There is a 3-cell K such that $J_1\subset\text{Int } K\subset K\subset\text{Int } S$, there is an annulus $B'\subset B$ such that $\text{Bd } R\subset\text{Bd } B'$ and $B'\cap K=\emptyset$, and there is a subarc F of J such that $p\in\text{Bd } F$, $F-p\subset\text{Int } S$, $F\cap K=\emptyset$, and $\text{Diam } F<\delta$. Using the hypothesis, we choose a polyhedral 2-sphere S' in Int S such that $S'\cap(J-A)\subset F$, $K\subset\text{Int } S'$, $S'\cap B\subset B'$, $S'\cap A$ is a point,

(1) $S'\subset N(S, \delta)$, and

(2) $d(x, S)<d(K, S)$ for all $x\in S'$.

We also assume that $S'\cap B'$ consists of a finite collection of disjoint simple closed curves $L_1, L_2, \cdots, L_n$ no one of which can be contracted in B. Then each L_i bounds a disk E_i on S' where $E_i\cap A=\emptyset$. We choose an integer k such that Int E_k contains no L_i. Then L_k and Bd R bound an annulus B'' on B' such that $B''\cup E_k$ is a disk D lying, except for its boundary Bd R, in Int S. Notice that $D\cap(K\cup A)=\emptyset$. In the next paragraph we complete the proof by showing diam $D<\epsilon$.

Let $E_k=E$. Since J links Bd E there exists a point $a\in J\cap\text{Int } E$. From the construction we see that $a\in F$, so $d(p, a)<\delta$. Suppose Diam E is greater than 6δ. Then some point $b\in\text{Int } E$ exists such that $d(a, b)>3\delta$ and there is an arc ab from a to b in Int E. It follows from (1) and (2) that there exists a point $c\in S$ such that the segment bc does not intersect $K\cup B\cup R$ and $d(,b\ c)<\delta$. The union of a subarc of F from p to a, ab, bc, and an open arc in Ext S from c to p is a simple closed curve J' such that $J'\cap B=\emptyset$, $J'\subset E^3-K$, and J' links Bd R. This means that J' links J_1 [**1**]; however, this is a contradiction since J_1 can be contracted in the 3-cell K. Hence Diam $E\leqq 6\delta$. Since Diam $B''<\delta$, we know that Diam $D<7\delta<\epsilon$.

THEOREM 2. *Suppose that S is a 2-sphere in E^3 such that the set W of points where S is wild from* Int S *is* 0*-dimensional. If for some point $q\in S$ there exists a homeomorphism h of* Int $S\cup q$ *onto* Int $B\cup b(b\in B)$, *then* $S\cup$ Int S *is a* 3*-cell.*

PROOF. It is easy to find an arc A' and a sequence $\{S_i'\}$ of round 2-spheres in the range of h such that their inverse images under h satisfy the hypothesis of Lemma 2. If $q\in S-W$, it follows from Lemma 2 that S is tame from Int S.

If $q\in W$ then S is locally tame from Int S at each point of $S-q$ [**5**]. That S is locally tame from Int S in this case follows from Theorem 1 and [**2**].

THEOREM 3. *If there exists a closed* 0*-dimensional subset G of a* 2*-sphere S such that, for each point $p\in S-G$, there exists a homeomorphism of $p\cup$*Int S *onto* $b\cup$Int B, *then* $S\cup$Int S *is a* 3*-cell.*

PROOF. Since $S-G$ is open it follows from Theorem 1 and [**2**] that S is locally tame from Int S at each point of $S-G$. Thus Theorem 3 follows from Theorem 2.

A point p of a 2-sphere S has been called a *piercing point of the crumpled cube* $C=S\cup$Int S if and only if there is a homeomorphism h of C into E^3 such that $h(S)$ can be pierced at $h(p)$ by a tame arc [**13**].

COROLLARY 3. *If S is a 2-sphere in E^3 such that, for each piercing point p of $S \cup \text{Int } S$, there exists a homeomorphism of* $\text{Int } S \cup p$ *onto* $\text{Int } B \cup b$ $(b \in B)$, *then* $S \cup \text{Int } S$ *is a 3-cell.*

PROOF. From the fact that $S \cup \text{Int } S$ has piercing points [**13**] and the hypothesis, we see that $\text{Int } S$ is an open 3-cell. Then $S \cup \text{Int } S$ has at most one nonpiercing point (see [**11**] or [**14**]), so Corollary 3 follows from Theorem 3.

THEOREM 4. *If there is a closed 0-dimensional subset G of a 2-sphere S such that for each $p \in S - G$ there exists a homeomorphism of $(E^3 - S) \cup p$ onto $(E^3 - B) \cup b$, then S is tame.*

PROOF. We may think of S as lying in S^3 so that it follows from Theorem 3 that S bounds two 3-cells in S^3. Thus S is tame.

COROLLARY 4. *If S is a 2-sphere in E^3 such that, for each point p of S that lies in a tame arc on S, there exists a homeomorphism of $(E^3 - S)$ $\cup P$ onto $(E^3 - B) \cup b$, then S is tame.*

PROOF. Corollary 4 follows from Theorem 4 because at most two points of S fail to lie in tame arcs in S (see [**8**] and [**11**] or [**14**]).

THEOREM 5. *Let S be a 2-sphere in E^3 and let $\epsilon > 0$. If for each $p \in S$ there exist an ϵ-disk D in S containing p and a homeomorphism h of* $\text{Int } S$ *onto* $\text{Int } B$ *such that h extends to take* $\text{Bd } D$ *homeomorphically into B, then* $S \cup \text{Int } S$ *is a 3-cell.*

PROOF. Let $p \in S$ and notice that $\text{Int } S$ is locally simply connected at p if p lies in the boundary of some disk D as given in the hypothesis (see Theorem 1). Otherwise p lies in the interiors of arbitrarily small such disks, and we are able to prove that S is locally α-capped from $\text{Int } S$ at p. Once this is done we rely on Lemma 1 and [**2**] to complete the proof.

Let U be an open set containing p, let D be a disk such that $p \in \text{Int } D \subset D \subset U \cap S$, and let h be a homeomorphism of $\text{Int } S$ onto $\text{Int } B$ that extends to take $\text{Bd } D$ homeomorphically into B. As in the proof of Theorem 1, we choose a disk E' in $h(U \cap \text{Int } S) \cup h(\text{Bd } D)$ such that $\text{Bd } E' = h(\text{Bd } D)$, and we let $E = h^{-1}(E')$. Such a disk E can be chosen in any neighborhood of S, so S is locally α-capped from $\text{Int } S$ at p.

THEOREM 6. *If for each point p of a closed subset F of a 2-sphere S in E^3 there exists a homeomorphism h of* $\text{Int } S \cup p$ *onto* $\text{Int } B \cup b$, *then F lies on the boundary of a 3-cell.*

PROOF. From Theorem 1 we see that Int S is locally simply connected at each point of F. Now the techniques in the proof of Theorem 1 in [2] can be applied, as indicated in the proof of Theorem 10 of [9], to yield Property (*, F, Int S); that is, S can be side approximated from Int S missing F (see [8] or [9] for a precise definition). The reason no restriction is placed on the size of the components of F in Theorem 6 is that the local simple connectedness of Int S at each point of F is a much stronger hypothesis than that of Theorem 10 of [9]. We omit the tedious details since they are like those of [9]. Once (*, F, Int S) is known the conclusion of Theorem 6 follows from [12].

THEOREM 7. *If S is a 2-sphere in E^3 and p is a point of S such that there exists a homeomorphism of* Int $S \cup p$ *onto* Int $B \cup b$ $(b \in B)$, *then p is a piercing point of* $S \cup$ Int S.

PROOF. As indicated in the proof of Theorem 6, Property (*, p, Int S) can be established. This is sufficient for p to be a piercing point of $S \cup$ Int S [11].

REMARK. Of course the converse to Theorem 7 is false even if Int S is an open 3-cell (see Theorem 2 and an appropriate example in [7]).

THEOREM 8. *Suppose F is a set as given in Theorem 6. If for each point p of F there exists a homeomorphism of $(E^3 - S) \cup p$ onto $(E^3 - B) \cup b$ $(b \in B)$, then F lies on a tame 2-sphere.*

PROOF. The proof here is much the same as that outlined for Theorem 6. We first obtain Property (*, F, S), then we appeal to Theorem 6 of [9].

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