A SEPARATION THEOREM FOR MANIFOLDS

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Let X be a compact connected n-manifold without boundary. We assume a cohomology theory satisfying the continuity axiom as well as the Eilenberg-Steenrod axioms and defined on a class of spaces which includes the compact manifolds, e.g., the Alexander theory on the category of compact pairs [**3**, Chapter 6, §4 et seq.]. The coefficient group throughout will be the additive group Z of integers if X is orientable and the group Z_2 of integers mod 2 if X is nonorientable. For standard results in the algebraic treatment of manifolds one may consult Eilenberg-Steenrod [**1**, Chapter 11, §6] or Spanier [**3**, Chapter 6]. We denote the q-dimensional cohomology group of X by $H^q(X)$; if $q=0$, $H^q(X)$ is taken to be the reduced zero-dimensional group.

For a closed proper subset A of X let $i: A \to X$ be the inclusion map. The purpose of this note is to prove the following theorem.

THEOREM 1. *The subset A separates X if and only if*

$$i^*: H^{n-1}(X) \to H^{n-1}(A)$$

is not surjective.

We shall need some preliminary notions and results; for details see Wallace [**4**] or Keesee [**2**].

The inclusion $i: A \to X$ induces homomorphisms $i^*: H^q(X) \to H^q(A)$; if $e \in H^q(X)$ we shall denote $i^*(e)$ by $e|A$, and if $u \in H^q(A)$ is in the image of i^* we shall say that u can be extended to X.

Let $e \in H^q(X)$ and let M be a closed proper subset of X such that $e|M \neq 0$ but $e|N=0$ for every closed proper subset N of M. Then M is called a ***minimal support*** for e.

Let A be a closed subset of X and $e \in H^q(A)$. An *irreducible membrane* for e is a closed subset R of X such that e cannot be extended to $R \cup A$ but can be extended to $S \cup A$ for each closed proper subset S of R.

We have the following results.

(i) There exists a minimal support for each nonzero element of $H^q(X)$. Each minimal support is connected.

(ii) If $e \in H^q(A)$ cannot be extended to X, then e has an irreducible membrane R in X. Furthermore, $e|(R \cap A) = e_0 \neq 0$, R is an irreducible

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membrane for e_0, $R=(R-A)^-$, and $R-A$ is connected.

(iii) If R is an irreducible membrane for an element in $H^{n-1}(A)$, then $R-A$ is open in X.

The proof of (iii) is similar to that in [2, Theorem 1.5].

The following lemma extends a familiar theorem for spheres to arbitrary manifolds. Recall that in a cohomology theory satisfying the continuity axiom on the category of compact pairs, every compact triad is proper and the associated Mayer-Vietoris sequence is exact.

LEMMA 1. *If A is a closed proper subset of the n-manifold X, then* $H^n(A)=0$.

PROOF. Suppose $e\in H^n(A)$, $e\neq 0$. Without loss of generality, we may assume that A is a minimal support for e.

Since A is not all of X, there is a point x in the boundary of A and we can find a coordinate nbd U of x whose boundary intersects $X-A$. The boundary of U, ∂U, is an S^{n-1}, and $\partial U\cap A=R$ is a closed proper subset of S^{n-1}; thus $H^{n-1}(R)=0$. Let $S=A-U$, $T=\overline{U}\cap A$; then $S\cap T=R$ and $S\cup T=A$. Consider the portion of the Mayer-Vietoris sequence of the triad $(A;S,T)$:

$$\cdots \to H^{n-1}(R)\to H^n(A)\xrightarrow{j} H^n(S)\oplus H^n(T)\to\cdots.$$

Since $H^{n-1}(R)=0$, j is injective. Thus $j(e)\neq 0$. But S and T are closed proper subsets of A, and since A is a minimal support for e, $e|S=e|T=0$, which implies that $j(e)=(e|S, e|T)=0$.

LEMMA 2. *If R is an irreducible membrane for an element in $H^{n-1}(A)$, then $R-A$ is a component of $X-A$.*

PROOF. Since $R-A$ is connected, it is contained in some component C of $X-A$. Suppose $R-A$ is not all of C. Then we have $C-R\neq\square$, $R-A\neq\square$, and $(R-A)\cup(C-R)=C$. Now $(R-A)^-=R$, so $(R-A)^-\cap(C-R)=\square$. Since $R-A$ is open in X, it follows that $(C-R)^-\cap(R-A)=\square$. But this implies that C is not connected.

We are now ready to prove the separation theorem.

THEOREM 2. *A separates X if and only if $i^*\colon H^{n-1}(X)\to H^{n-1}(A)$ is not surjective.*

PROOF. (a) Suppose A separates X; $X-A=P\cup Q$, P is separated from Q, P and Q nonempty. Let $R=A\cup P$, $S=A\cup Q$; then $R\cap S=A$ and $R\cup S=X$. Consider the portion of the Mayer-Vietoris sequence for $(X;R,S)$:

$$\cdots \to H^{n-1}(R) \oplus H^{n-1}(S) \xrightarrow{i'} H^{n-1}(A) \xrightarrow{\delta} H^n(X) \xrightarrow{j'} H^n(R) \oplus H^n(S) \to \cdots.$$

Now $i^*: H^{n-1}(X) \to H^{n-1}(A)$ surjective would imply i' surjective, $\delta = 0$, and j' injective. This is impossible since $H^n(X) \neq 0$ and $H^n(R) \oplus H^n(S) = 0$.

(b) Now suppose $X - A$ is connected, and assume that i^* is not surjective. Then there is an element $e \in H^{n-1}(A)$ that cannot be extended to X; e has an irreducible membrane R in X.

By Lemma 2, $R - A$ is a component of $X - A$ and thus $R - A = X - A$. This implies that $R \cup A = X$.

Let U be a coordinate nbd in $R - A$ such that $\overline{U}$ does not intersect A. Now, since R is an irreducible membrane for e, e can be extended to an element

$$\bar{e} \in H^{n-1}(A \cup (R - U)) = H^{n-1}(X - U).$$

The boundary of U is an S^{n-1}; and $\bar{e} | S^{n-1} \neq 0$, for if it were, e could be extended to $R \cup U$ (see [2, Theorem 1.2]). Consider the portion of the Mayer-Vietoris sequence for $(X; \overline{U}, X - U)$:

$$\cdots \to H^{n-1}(\overline{U}) \oplus H^{n-1}(X - U) \xrightarrow{i'} H^{n-1}(S^{n-1}) \xrightarrow{\delta} H^n(X) \to H^n(\overline{U}) \oplus H^n(X - U) \to \cdots.$$

Since the last group is trivial, δ is surjective. Thus

$$H^{n-1}(S^{n-1})/\operatorname{Im} i' \approx H^n(X).$$

Now $\operatorname{Im} i' \neq 0$, since we know that $\bar{e} | S^{n-1} \neq 0$. If X is orientable the isomorphism above becomes $Z_n \approx Z$ for some $n \neq 1$; if X is nonorientable it becomes $H^n(X) = 0$. In either case we have a contradiction.

COROLLARY 1. *The subset A separates X if and only if*

$$j^*: H^n(X, A) \to H^n(X)$$

is bijective.

We also obtain the following well-known characterization of sets that separate spheres [2].

COROLLARY 2. *A closed proper subset A of S^n separates S^n if and only if $H^{n-1}(A) \neq 0$.*

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