

COMMUTATIVE RIMS IN CLANS WITH ZERO

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A semigroup is a Hausdorff space with a continuous, associative multiplication defined on it. Standard references are [1] and [2]. No notational distinction will be made between the semigroup and the underlying space. If S is a compact semigroup and if $e^2 = e \in S$, then the (unique) maximal group containing e will be denoted $H(e)$. The (unique) minimal ideal of S will be denoted $M(S)$. A hormos is defined in [1]. For our purposes it is sufficient to know that a hormos is a compact, connected, and abelian semigroup with identity. This paper relies heavily on the Second Fundamental Theorem of [1] and we state the pertinent parts:

SECOND FUNDAMENTAL THEOREM. *Let S be a compact semigroup. Then the following properties are equivalent:*

(1) *The connected component of each idempotent meets the minimal ideal.*

(2) *If e is an idempotent, there is an irreducible hormos T in eSe such that $T \cap H(e) = \{e\}$ and $T \cap M(S) \neq \emptyset$.*

(3) *If e is an idempotent and A_e a compact, connected, abelian subgroup of $H(e)$, then there is an irreducible hormos T in the centralizer of A_e and in eSe such that $T \cap H(e) = \{e\}$ and $T \cap M(S) \neq \emptyset$.*

Throughout this paper cohomology is that of Alexander-Spanier with coefficient group arbitrary.

DEFINITION. Let X be a space and R a compact subset of X . Then R is called a *rim* of X if for each proper closed subset Y containing R there is some integer n such that the map $H^n(X) \rightarrow H^n(Y)$ induced by inclusion is not onto. (See [3], but note the slight difference.)

LEMMA. *Let S be a continuum semigroup containing 0 and 1 and let R be a rim of S . If T is a continuum subsemigroup containing 0 and 1, then $S = RT$.*

PROOF. It is clear that $R \subseteq RT$ and that RT is homotopic to 0. Hence $H^n(S) \rightarrow H^n(RT)$ is onto for all n . Since R is a rim, $RT = S$. This lemma admits a quick generalization to actions.

THEOREM. *Let S be a continuum semigroup with 0 and 1 and let R be a rim of S . Assume R is connected and if $x, y \in R$, then $xy = yx$. Then S is commutative.*

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PROOF. It is known [3] that $1 \in R$. Let C be the smallest compact semigroup in S containing R . C is a continuum subsemigroup, $1 \in C$, and C is abelian. Therefore if $e \in M(C)$, then $M(C) = H(e) \cap C$. Also $eSe \cap C = H(e) \cap C$.

By the SFT there is a hormos I in C from 1 to $H(e) \cap C$. There is also a hormos T in eSe from e to 0 and $T \cap H(e) = \{e\}$. We may choose T in the centralizer of $M(C)$ so that $M(C)T$ is a semigroup. Relying on the lemma we have $R(I \cup M(C)T) = S$. Now $RI \subseteq C$ so $eSe \subseteq RM(C)T$. But $RM(C)T \subseteq M(C)T \subseteq H(e)T \subseteq eSe$. Thus $eSe = RM(C)T = M(C)T$ an abelian semigroup.

Now $S = C \cup eSe$ and both members of the union are abelian. Let $c \in C$ and $d \in eSe$. Then $cd = ced = dce = dece = dec = dc$. Hence members of C and eSe commute with each other and S is abelian.

EXAMPLE. Let $S_1 = \{(a, b) \mid (a, b) \in R_2, |a| + |b| \leq 1\}$. Define a multiplication on S_1 by $(a, b)(c, d) = (|ac|, |ad| + |b|)$. $M(S_1) = \{(0, x) \mid 0 \leq x \leq 1\}$. $S_1/M(S_1)$ (the Rees quotient) is a compact semigroup with 0 and connected, commutative rim. However, $S_1/M(S_1)$ is not commutative.

Let S_2 be $[0, 1]$ with the usual multiplication.

Take the disjoint union of $S_1/M(S_1)$ and S_2 and identify their respective zeros. If $(a, b) \in S_1/M(S_1)$ and $x \in S_2$, define

$$(a, b)x = (ax, b),$$

$$x(a, b) = (ax, bx).$$

The space S so defined is a continuum semigroup with zero and 1 . It has for a rim the rim of $S_1/M(S_1)$ together with the identity. The rim is commutative, but S is not.

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