## SHORTER NOTES

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# BINARY RELATIONS ON SETS OF REGULAR CARDINALITY

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1. **Introduction.** In [5, Theorem 2.1] the second author showed that a lattice of regular cardinality $m$ contains either a proper principal ideal of cardinality $m$, a proper principal dual ideal of cardinality $m$, or a two-dimensional sublattice of cardinality $m$. In this note the proof of the above mentioned theorem is modified to extend the result to arbitrary binary relations on a set of regular cardinality. Thanks are due to B. Jónsson for a useful suggestion in the formulation of the theorem.

2. **Notation and definitions.** Let $X$ be a set and let $R$ be a binary relation of $X$; that is $R \subset X \times X = X^2$. For $x \in X$ define $(x)$ and $[x]$ as follows:

$$(x) = \{y \in X \mid (y, x) \in R\}, \quad \text{and} \quad [x] = \{y \in X \mid (x, y) \in R\}.$$

A pair of elements $x$ and $y$ are called *complements* if $z \in [x] \cap [y]$ implies $(z) = X$ and $w \in (x) \cap (y)$ implies $[w] = X$.

Denoting the cardinal number of a set $A$ by $|A|$, we recall that a cardinal $m$ is called *regular* if it is impossible to have $m = \sum_{i \in I} m_i$, where $|I| < m$ and for each $i \in I$, $m_i < m$.

A subset $Y$ of $X$ is called an *antichain* if every pair of distinct elements of $Y$ are incomparable relative to $R$; that is, $x, y \in Y$ and $x \neq y$ implies $(x, y) \notin R$. The set $Y$ is called a *chain* if every pair of distinct elements are comparable; given distinct $x$ and $y$ in $Y$, either $(x, y) \in R$ or $(y, x) \in R$.

3. **The principal result.** We now state and prove the principal result of this note.

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THEOREM. *If $X$ is a set of regular cardinality $m$ and if $R$ is a proper binary relation on $X$, then one of the following conditions must hold*:

(i) *there is an $x \in X$ such that $(x) \neq X$ and $|(x)| = m$,*

(ii) *there is an $x \in X$ such that $[x] \neq X$ and $|[x]| = m$,*

(iii) *there is an antichain $U \subset X$ with $|U| = m$ such that any two distinct elements of $U$ are complements.*

PROOF. Assume that (i) and (ii) fail. Since $R \neq X^2$ there is some $x \in X$ for which $(x) \neq X$. Now $A = \{y \in X \mid yRz \text{ for all } z \in X\} \subset (x)$, and since $|(x)| < m$, we must have $|A| < m$. Similarly if $B = \{y \in X \mid zRy$ for all $z \in X\}$, we have $|B| < m$, and thus $|A \cup B| < m$.

Now for $x \in X - (A \cup B)$ we let $C(x) = \{y \in X \mid y$ is a complement of $x$, $(y, x)$ and $(x, y) \notin R$, and $x \neq y\}$.

If $y \notin C(x)$ then one of the following must hold:

$y \in (z)$ where $z \in [x]$ and $z \notin A$;

$y \in [z]$ where $z \in (x)$ and $z \notin B$;

$y \in (x) \cup [x]$; $y = x$.

Therefore we have

$$X - C(x) \subset [\bigcup \{(z) \mid z \in [x]\}] \\ \cup [\bigcup \{[z] \mid z \in (x)\}] \cup (x) \cup [x] \cup \{x\},$$

and using the regularity of $m$ we see that $|X - C(x)| < m$. The preceding inequality is thus true for any $x \notin A \cup B$.

Suppose $\beta < m$ and $\{x_\xi \mid \xi < \beta\}$ is such that for $\xi, \xi' < \beta$ and $\xi \neq \xi'$ we have $x_\xi \in C(x_{\xi'})$. Now

$$\begin{aligned} \left| X - \bigcap \{C(x_\xi) \mid \xi < \beta\} \right| &= \left| \bigcup \{(X - C(x_\xi)) \mid \xi < \beta\} \right| \\ &\leqq \sum \{ |X - C(x_\xi)| \mid \xi < \beta\} \\ &< m. \end{aligned}$$

We now take $x_\beta \in \bigcap \{C(x_\xi) \mid \xi < \beta\}$ and by appeal to the principle of transfinite induction obtain $U = \{x_\xi \mid \xi < m\}$ having the desired properties.

REMARK. The necessity of the regularity assumption has been demonstrated in [5].

4. **Some observations.** In this section we consider some particularizations of the theorem in §2 of actual and potential utility.

*Observation* 1. It is clear that if $R$ is a lattice ordering, then the theorem of [5] cited in the introduction follows from the theorem of this note. In fact, it is sufficient to assume that $R$ is reflexive, antisymmetric and both up- and down-directed.

*Observation* 2. If $T$ is a semigroup, or more generally, $T$ is a semigroup "acting" on a set $X$; that is, there is given a function $T\times X\rightarrow X$ satisfying $t_1(t_2x)=(t_1t_2)x$ for all $t_1t_2\in T$ and for each $x\in X$, the quasi-order $Q$ on $X$ defined by

$$Q = \{(x, y) \mid x \in Ty \cup \{y\}\},$$

has played a role in the structural analysis of semigroups and acts (automata). (See [1] and its bibliography.) It is not our intention to consider herein an application of our theorem to the above-mentioned structures but only to hint at its possible utility when the set $X$ has regular cardinality.

*Observation* 3. It was noted in [2] that every countable partially ordered set contains an infinite chain or infinite antichain. This, as well as the more general proposition with the partial order replaced by an arbitrary relation, is a direct consequence of a theorem of F. P. Ramsey [3]. It is of interest to note that the observation in [2] as well as the extension mentioned above follow directly from the theorem of this note. For, if $R$ is a binary relation (which we can without loss of generality take to be reflexive and symmetric) on a countable set $X$, then either (iii) of our theorem holds giving the infinite antichain, or the theorem can be applied repeatedly to the sets $(x)$ given by (i) to generate an infinite chain.

## References


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