THE UNIVERSAL COMPACT SUBUNITHETIC SEMIGROUP

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Compact subunithetic semigroups have been studied in [3] and [4], and related results can be found in [5], [6], and [7].

The structure of compact subunithetic semigroups is completely determined in this paper by exhibiting a universal compact subunithetic semigroup in the continuous homomorphism sense and a universal compact unithetic semigroup in the embedding sense. Generalizations of some of the results of [4] to include nonabelian semigroups are obtained.

If S is a compact [uniquely] divisible semigroup and $x \in S$, then there exists a [unique] minimal compact divisible subsemigroup $S(x)$ of S which contains x. Moreover, each such $S(x)$ is subunithetic [unithetic]. Thus the study of the structure of compact subunithetic semigroups is essential to the study of compact divisible semigroups.

NOTATION. The following notation will be used throughout this paper:

1. N = set of all positive integers;
2. Q = discrete additive semigroup of positive rationals;
3. $I = [0, 1]$ with usual multiplication and topology;
4. $\Sigma = a$-adic solenoid with $a = (2, 3, \cdots)$ [2, p. 114];
5. Σ^* = universal compact solenoidal group [2, 25.19];
6. Φ = universal compact solenoidal semigroup [6, II].

A semigroup S is said to be [*uniquely*] *divisible* if for each $y \in S$ and each $n \in N$, there exists an [unique] element $x \in S$ such that $y = x^n$. A topological semigroup T is said to be *subunithetic* if T contains a dense homomorphic image of Q (Note that T is divisible and abelian). A subunithetic semigroup T is said to be *unithetic* if T is uniquely divisible. If T is a unithetic semigroup and $\sigma: Q \to T$ is a homomorphism such that $\sigma(Q)$ is dense in T, then the element $x = \sigma(1)$ is called a *unithetic generator* of T. (Note that the rational powers of x are dense in T.)

If S is a uniquely divisible topological semigroup and $x \in S$, then the subsemigroup $[x] = \{x^r : r \in Q\}^-$ (closure in S) is the unithetic subsemigroup of S generated by x. Note that S is unithetic if and only if $S = [x]$ for some $x \in S$.

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The *Bohr* (or *almost periodic*) *compactification* of a topological semigroup S is a pair (B, β) such that:

(i) B is a compact semigroup;

(ii) $\beta: S \to B$ is a continuous homomorphism of S into B such that $\beta(S)$ is dense in B; and

(iii) if $f: S \to T$ is a continuous homomorphism of S into a compact semigroup T, then there exists a unique continuous homomorphism $f^*: B \to T$ such that $f^*\beta = f$.

The existence and uniqueness of the Bohr compactification can be obtained as a consequence of the adjoint functor theorem [**8**]. Related results can be found in [**1**]. The semigroup Φ is the Bohr compactification of the additive semigroup R of nonnegative reals with the usual topology, and Σ^* is the Bohr compactification of the group $R \cup (-R)$. Both Φ and Σ^* are unithetic. One of the most essential results in this paper is that $\Sigma^* \times \Phi$ is the Bohr compactification of Q.

LEMMA 1. *Let $h \in \Sigma^*$ and $\{p_n\}$ a sequence of positive prime integers such that $(n+1)! < p_n$ and $p_n + n < p_{n+1}$ for each $n \in N$. Then there exists $g \in \Sigma^*$ such that $\{g^{1/p_n}\}$ converges to h.*

PROOF. We first prove this result for Σ. Note that Σ is the projective limit of copies of the circle group with bonding sequence $(2, 3, 4, \cdots)$. Let $h_0 = (h_1, h_2, \cdots)$ be an element of Σ. We will construct $g_0 = (g_1, g_2, \cdots)$ in Σ inductively such that $\{g_0^{1/p_n}\}$ converges to h_0. Let $g_1 = \cdots = g_{p_1-1} = \exp(2\pi i)$. Let $n \in N$, and suppose $g_{p_n-1} = \exp(2\pi i S_n)$ has been defined for some real number S_n. Now $h_n = \exp(2\pi i t_n)$ for some real number t_n. Then, since $h_0 \in \Sigma$, there exist real numbers $t_1, t_2, \cdots, t_{n-1}$ such that $(j+1)!t_{j+1} = t_1$ and $h_j = \exp(2\pi i t_j)$ for $j = 1, 2, \cdots, h-1$. Let

$$r_n = \frac{t_1 p_n - S_n(p_n - 1)!}{n! p_n}.$$

Then there exists an integer k_n such that $p_n r_n - 1 \leqq -k_n \leqq p_n r_n + 1$. Since p_n is prime, $((p_n - 1)! + 1)/p_n$ is an integer, and hence $m_n = k_n((p_n - 1)! + 1)/p_n$ is an integer. Thus

$$t_1 - n!/p_n \leqq (S_n + h!\ k_n) \cdot (p_n - 1)!/p_n - h! m_n \leqq t_1 + n!/p_n.$$

It now follows that

$$t_{j+1} - \frac{n!}{p_n(j+1)!} \leqq \frac{(S_n + n!k_n)(p_n - 1)!}{p_n(j+1)!} - \frac{n! m_n}{(j+1)!}$$
$$\leqq t_{j+1} + \frac{n!}{p_n(j+1)!}$$

and $n!\,m_n/(j+1)!$ is an integer for $j=0, 1, 2, \cdots, n-1$. Define

$$g_{p_n+j} = \exp\left(\frac{2\pi i(S_n + n!k_n)(p_n - 1)!}{(p_n + j)!}\right) \qquad \text{for } j = 0, 1, \cdots, n-1,$$

and define g_j such that $g_j^j = g_{j-1}$ for $j = p_n+n, \cdots, p_{n+1}-1$. Thus g_0 is defined inductively such that $g_0 \in \Sigma$. Now

$$g_0^{1/p_n} = (g_{p_n}^{p_n!/p_n}, \cdots, g_{p_n+j}^{(p_n+j)!/p_n(j+1)!}, \cdots)$$

and

$$g_{p_n}^{(p_n+j)!/p_n(j+1)!} = \exp\left(\frac{2\pi i(S_n + n!k_n)(p_n - 1)!}{p_n(j + 1)!}\right)$$

$$\text{for } j = 0, 1, \cdots, n-1.$$

Since $\{n!/p_n\}$ converges to 0, it follows from the last inequality above that $\{g_0^{1/p_n}\}$ converges to h_0. This proves the result for Σ.

Let $\Sigma_\alpha = \Sigma$ for $\alpha \in I$. Then $\Sigma^* = \prod\{\Sigma_\alpha : \alpha \in I\}$. Let $\pi_\alpha : \Sigma^* \to \Sigma_\alpha$ be the projection map, and $h_\alpha = \pi_\alpha(h)$ for each $\alpha \in I$. Then there exists $g_\alpha \in \Sigma_\alpha$ such that $\{g_\alpha^{1/p_n}\}$ converges to h_α for each $\alpha \in I$. Let $g \in \Sigma^*$ such that $\pi_\alpha(g) = g_\alpha$ for each $\alpha \in I$. Then $\{g^{1/p_n}\}$ converges to h, and the proof of the lemma is complete.

THEOREM 2. *The Bohr compactification of Q is $\Sigma^* \times \Phi$.*

PROOF. Let x be a unithetic generator of Φ and 1 the identity of Φ. Let h be a unithetic generator of Σ^* and $\{p_n\}$ a sequence of positive prime integers such that $(n+1)! < p_n$ and $p_n + h < p_{n+1}$ for each $n \in N$. Then, by Lemma 1, there exists $g \in \Sigma^*$ such that $\{g^{1/p_n}\}$ converges to h. Thus $\{(g, x)^{1/p_n}\}$ converges to $(h, 1)$. Since h is a unithetic generator of Σ^*, it follows that $\Sigma^* \times \{1\} \subset [(g, x)]$. Let $(a, b) \in \Sigma^* \times \Phi$. Then there exists $u \in \Sigma^*$ such that $(u, b) \in [(g, x)]$. Let u^{-1} denote the inverse of u in Σ^*. Then $(u^{-1}a, 1) \in [(g, x)]$ and hence $(a, b) = (u, b) \cdot (u^{-1}a, 1)$ is in $[(g, x)]$. It follows that $\Sigma^* \times \Phi$ is unithetic, and that (g, x) is a unithetic generator of $\Sigma^* \times \Phi$. Define $\beta : Q \to \Sigma^* \times \Phi$ by $\beta(r) = (g, x)^r$ for each $r \in Q$. Then β is a homomorphism such that $\beta(Q)$ is dense in the compact semigroup $\Sigma^* \times \Phi$. Let S be a compact subunithetic semigroup, e the identity of S, and $f : Q \to S$ a homomorphism such that $f(Q)$ is dense in S. Then S contains a compact solenoidal subsemigroup T such that $S = H(e)T$ [**4**, Theorem 2.2]. Define $\phi : H(e) \times T \to S$ by $\phi(s, t) = st$; $s \in H(e)$, $t \in T$. Then ϕ is a continuous onto homomorphism since S is abelian. Let $s_0 \in H(e)$ and $t_0 \in T$ such that $\phi(s_0, t_0) = f(1)$. Then there exist continuous onto homomorphisms $\alpha : \Sigma^* \to H(e)$ and $\lambda : \Phi \to T$ such that $\alpha(g) = s_0$ and $\lambda(x) = t_0$ [**4**]. Define $f_1^* : \beta(Q) \to S$ by $f_1^*\beta(r) = \phi(\alpha(g^r), \lambda(x^r))$; $r \in Q$. Then, since $\beta(Q)$

is dense in $\Sigma^*\times\Phi$, f_1^* has a unique extension to $f^*:\Sigma\times\Phi\rightarrow S$, and $f^*\beta=f$. It follows that $(\Sigma^*\times\Phi, \beta)$ is the Bohr compactification of Q.

COROLLARY 3. *A compact semigroup is subunithetic if and only if it is a continuous homomorphic image of* $\Sigma^*\times\Phi$.

COROLLARY 4. *Let S and T be compact semigroups. Then $S\times T$ is [sub-] unithetic if and only if S and T are [sub-] unithetic and either S or T is a group.*

COROLLARY 5. *The semigroup* $((\Sigma^*\times I)/(\Sigma^*\times\{0\}))\times\Sigma^*$ *is unithetic.*

THEOREM 6. *Let S be a compact semigroup. Then these are equivalent:*
(i) *S is [uniquely] divisible:*
(ii) *each component of S is a [uniquely] divisible subsemigroup of S; and*
(iii) *if $x\in S$, then there exists a [unique] subsemigroup $S(x)$ of S, which is minimal with respect to being a compact divisible subsemigroup of S containing x; and each such $S(x)$ is [unithetic] subunithetic.*

PROOF. The proof follows from the fact that $\Sigma^*\times\Phi$ is connected, and, if S is divisible, and $x\in S$, then there exists a continuous homomorphism λ: $\Sigma^*\times\Phi\rightarrow S$ such that $\lambda(y)=x$, for some unithetic generator y of $\Sigma^*\times\Phi$.

NOTATION. If S is a semigroup, then $E(S)$ denotes the set of idempotent elements of S.

COROLLARY 7. *Let S be a compact totally disconnected semigroup. Then S is divisible if and only if $S=E(S)$.*

COROLLARY 8. *A finite semigroup S is divisible if and only if $S=E(S)$.*

COROLLARY 9. *Let S be a compact semigroup. Then each element of $S\backslash E(S)$ lies on a [unique usual] thread in S if and only if S is [uniquely] divisible and has degenerate subgroups.*

THEOREM 10. *Let S be a compact unithetic semigroup. Then S is topologically isomorphic to a subsemigroup of* $((\Sigma^*\times I)/(\Sigma^*\times\{0\}))\times\Sigma^*$.

PROOF. Let 1 denote the identity of S, K the minimal ideal of S, and h: $H(1)\rightarrow\Sigma^*$ and $k:K\rightarrow\Sigma^*$ injections. (See [3, Theorem 2.3 and Theorem 3.1].) Let γ: $S/K\rightarrow(H(1)\times I)/(H(1)\times\{0\})$ be a topological isomorphism. (See [3, Theorem 3.4].) Let j: $I\rightarrow I$ be the identity map, α: $H(1)\times I\rightarrow(H(1)\times I)/(H(1)\times\{0\})$ the natural maps, and $\lambda:\Sigma^*\times I\rightarrow(\Sigma^*\times I)/(\Sigma^*\times\{0\})$ the natural map. Then there exists an induced injection ϕ such that the diagram:

$$\begin{array}{ccc}(H(1)\times I)/(H(1)\times\{0\}) & \xrightarrow{\phi} & (\Sigma^*\times I)/(\Sigma^*\times\{0\}) \\ \alpha\uparrow & & \uparrow\lambda \\ H(1)\times I & \xrightarrow{h\times j} & \Sigma^*\times I\end{array}$$

commutes. Let e denote the identity of K and define $\sigma: S\to(S/K)\times K$ by $\sigma(x)=(\psi(x), ex)$; $x\in S$, where ψ: $S\to S/K$ is the natural map. Then σ is an injection. Let $\rho: \Sigma^*\to\Sigma^*$ be the identity map. Then

$$S\xrightarrow{\sigma}(S/K)\times K\xrightarrow{\gamma\times k}\frac{(H(1)\times I)}{(H(1)\times\{0\})}\times\Sigma^*\xrightarrow{\phi\times\rho}\frac{(\Sigma^*\times I)}{(\Sigma^*\times\{0\})}\times\Sigma^*$$

defines the desired injection, and the proof of the theorem is complete.

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