

SYMBOLIC CALCULUS FOR ALGEBRAS OF FOURIER-STIELTJES TRANSFORMS

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The purpose of this note is to prove Theorem 1 below. G is a non-discrete locally compact abelian group with dual group Γ . The algebra of Fourier transforms on Γ is $A(\Gamma)$, and the algebra of Fourier-Stieltjes transforms is $B(\Gamma)$. If μ is a measure in G , $\hat{\mu}$ is its Fourier-Stieltjes transform, and $\|\hat{\mu}\|$ is the total variation norm $\|\mu\|$.

THEOREM 1. *Let X be a closed subalgebra of $B(\Gamma)$ such that*

- (i) $A(\Gamma) \subseteq X$,
- (ii) $\delta_0 \in X$ (δ_0 is the identity of $B(\Gamma)$),
- (iii) X contains a real function which is the transform of a measure whose powers are pairwise mutually singular.

Suppose F is a complex valued function defined on $[-1, 1]$ and suppose $F \circ f \in B(\Gamma)$ for all $f \in X$ such that $f(\Gamma) \subseteq [-1, 1]$. Then F is the restriction to $[-1, 1]$ of an entire function.

Once Lemma 6.3.6 of [2] is extended to our version (Lemma 2 below), the proof of Theorem 1 follows the pattern of [2, 6.3]. We shall let $F \circ \mu$ be the unique measure having $F \circ \hat{\mu}$ as its transform (when $F \circ \mu$ is defined).

LEMMA 1 [2, 2.6.7]. *Let C be a compact subset of Γ , Γ a locally compact group and $\epsilon > 0$. Then there is a Fourier transform $f \in A(\Gamma)$ such that*

- (i) $\|f\| \leq 1 + \epsilon$,
- (ii) support f is compact,
- (iii) f is one on C , and $0 \leq f \leq 1$.

LEMMA 2. *Suppose F is defined on all of the line R , and X satisfies the hypotheses of Theorem 1, and $F \circ f \in B(\Gamma)$ whenever f is a real function in X . Then for each real function $\hat{\mu} \in X$ there exist numbers $\eta > 0$, K such that $\nu \in X$ a real function and $\|\nu\| < \eta$ imply*

$$(1) \quad \|F \circ (\mu + \nu)\| < K.$$

PROOF. If the conclusion is false, then there exist μ and $\nu_j \in X$ such that $\|\nu_j\| < 2^{-j}$, $\hat{\mu}(\Gamma) \subset R$, $\hat{\nu}_j(\Gamma) \subseteq R$, and $\|F \circ (\mu + \nu_j)\| > j$, $j = 1, 2, 3, \dots$. We may, in fact, by applying Lemma 1 and [2, 1.9.1] assume that $\nu_j \in L^1(G)$ and $\hat{\nu}_j$ has compact support. Indeed,

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$\|F \circ (\mu + \nu_j)\| > j$ implies by [2, 1.9.1] that there exists a trigonometric polynomial

$$(2) \quad P(x) = \sum_{k=1}^N c_k(x, \gamma_k), \quad \gamma_k \in \Gamma$$

with

$$(3) \quad \|P\|_\infty = \sup\{ |P(x)| : x \in G \} \leq 1$$

and

$$(4) \quad \left| \sum_k c_k F \circ ((\hat{\mu} + \hat{\nu}_j)(\gamma_k)) \right| > j.$$

Let $q \in L^1(G)$ be chosen by Lemma 1 so that $\|q\|_1 < 2$ and $\hat{q} = 1$ on support $\hat{P} (= \{\gamma_k\}_{k=1}^N)$. Then $\|F \circ (\mu + q * \nu_j)\| > j$.

The basic trick now is to “adjust” the ν_j so that the supports of their transforms are “very” disjoint. That way we can add up the ν_j (since $\|\nu_j\| < 2^{-j}$ we get an element of $L^1(G)$), $f = \sum \nu_j$, and then pick out $F \circ (\mu + \nu_j)$ from $F \circ (\mu + f)$ by multiplying the latter (roughly speaking) by a function $h_j \in L^1(G)$ such that $\hat{h}_j = 1$ on the support of $\hat{\nu}_j$, but $\hat{h}_j \cdot \hat{h}_k = 0$ if $j \neq k$. We then will get an equality of the form

$$(5) \quad h_j * F \circ (\mu + f) + (\delta_0 - h_j) * F \circ (\mu) = F \circ (\mu + \nu_j).$$

Now take norms and apply the triangle inequality and the multiplication inequality. We obtain

$$(6) \quad \|h_j\| \|F \circ (\mu + f)\| + \|(\delta_0 - h_j)\| \|F \circ (\mu)\| \geq \|F \circ (\mu + \nu_j)\| > j.$$

The h_j will be chosen with $\|h_j\| < 6$, so we have the absurdity:

$$(7) \quad 6\|F \circ (\mu + f)\| + 7\|F \circ (\mu)\| > j \quad \text{for each } j.$$

Before we give the details of the adjustments we must make to the ν_j , we first observe that if the supports of the ν_j all lie in the same compact subset of Γ , then $\|F \circ (\mu + \nu_j)\|$ must be bounded. Indeed, let support $\hat{\nu}_j \subset C$ for all j . Let $g, h \in L^1(G)$ be such that support \hat{g} and support \hat{h} are compact, $\|g\| < 2$, $\|h\| < 2$, $\hat{g} = 1$ on C and $\hat{h} = 1$ on support \hat{g} . Then $F \circ (\mu + \nu_j) = g * F \circ (\mu + \nu_j) + (\delta_0 - g) * F \circ (\mu + \nu_j) = g * F \circ (h * \mu + \nu_j) + (\delta_0 - g) * F \circ \mu$. We now need only show that the term $F \circ (h * \mu + \nu_j)$ is norm-bounded.

Case I. Γ discrete. We may use the arguments of [2, 6.3.4] to show that F is continuous. Let N be the number of elements of $C' = \text{support } \hat{h}$. Then N is finite. Let $L = \sup\{ |F(z)| : z \in (\hat{\mu} + \hat{\nu}_j)(C'), j = 1, 2, \dots \}$.

L is also finite since the sup is over a relatively compact set inside domain F . Then

$$(8) \quad \|F \circ (h * \mu + \nu_j)\| \leq NL.$$

This follows because $h * \mu + \nu_j$, and $F \circ (h * \mu + \nu_j)$ are trigonometric polynomials and G is compact.

Case II. Γ nondiscrete. Then by [2, 6.5.1 and 6.6.1], we may assume F is analytic in a neighborhood of R . Now, $h * \mu + \nu_j \in L^1(G)$. The (Cauchy) formula of Gelfand [1, Theorem 19] applies. Thus $F \circ (h * \mu + \nu_j)$ is norm-bounded.

We are now ready to adjust the ν_j . We will obtain functions g_j and $h_j \in L^1(G)$ such that (i) $\|g_j\|_1 < 2^{-j}$, (ii) support $\hat{g}_j \subseteq \{\hat{h}_j = 1\}$, (iii) $0 \leq \hat{h}_j \leq 1$, (iv) $\|h_j\|_1 < 6$, (v) support \hat{h}_j is compact, (vi) $\|F \circ (\mu + g_j)\| > j$ and (vii) $h_j * h_k = 0$ if $j \neq k$. We will construct these functions by induction. Let $g_1 = \nu_1$, and pick any $h_1 \in L^1(G)$ satisfying (ii)–(v). One exists by [2, 2.6.7]. Now assume we have found g_1, \dots, g_n and h_1, \dots, h_n satisfying (i)–(vii) for $j, k \leq n$. We construct g_{n+1} and h_{n+1} .

Pick $a, b, c \in L^1(G)$ such that $0 \leq \hat{a}, \hat{b}, \hat{c}, \leq 1$, $\|a\|_1, \|b\|_1, \|c\|_1 < 2$, and support $\hat{h}_j \subseteq \{\hat{a} = 1\} \subseteq \text{support } a \subseteq \{\hat{b} = 1\} \subseteq \text{support } b \subseteq \{\hat{c} = 1\}$, and support \hat{c} is compact. This can be done by repeated application of Lemma 1.

We claim $\|F \circ (\mu + (\delta_0 - a) * \nu_j)\| \rightarrow \infty$ as $j \rightarrow \infty$. Indeed,

$$(9) \quad \begin{aligned} F \circ (\mu + \nu_j) &= b * F \circ (\mu + \nu_j) + (\delta_0 - b) * F \circ (\mu + \nu_j) \\ &= b * F \circ (c * \mu + c * \nu_j) \\ &\quad + (\delta_0 - b) * F \circ (\mu + (\delta_0 - a) * \nu_j). \end{aligned}$$

The first term on the right-hand side of (9) is norm-bounded since support $\hat{c} \nu_j \subseteq \text{support } \hat{c}$ (which was chosen compact) for all j . Therefore, $\|F \circ (\mu + (\delta_0 - a) * \nu_j)\|$ is increasing. Pick j so large that $\|F \circ (\mu + (\delta_0 - a) * \nu_j)\| > n + 1$ and such that $\|\nu_j\| < 2^{-(n+3)}$. Let $g_{n+1} = (\delta_0 - a) * \nu_j$. Then support \hat{g}_{n+1} is compact, so there exists $f \in L^1(G)$ such that $0 \leq \hat{f} \leq 1$, $\|f\|_1 < 2$, support \hat{f} compact and $\hat{f} = 1$ on support \hat{g}_{n+1} . Let $h_{n+1} = (\delta_0 - a) * f$. Then g_1, \dots, g_{n+1} and h_1, \dots, h_{n+1} satisfy (i)–(viii) above. This completes the induction.

Now set $g = \sum_{n=1}^{\infty} g_n$. Then

$$(10) \quad \begin{aligned} F \circ (\mu + g_n) &= (\delta_0 - h_n) * F \circ (\mu + g_n) + h_n * F \circ (\mu + g_n) \\ &= (\delta_0 - h_n) * F \circ \mu + h_n * F \circ (\mu + g). \end{aligned}$$

This now yields the absurdity of (7).

Examples of algebras X which satisfy (i)–(iii) are easily given: Let E be a perfect compact independent subset of G . Such a set exists if G is not discrete [2, 5.2.2]. Let X be the closed subalgebra of $B(\Gamma)$ which is generated by $A(\Gamma)$, δ_0 , and the transforms of continuous measures supported on $E \cup -E$. One need only apply [2, 5.3.2] to show that X satisfies (iii).

Varopoulos [3] has shown that if G is not discrete, then only entire functions operate on $B_0(\Gamma) = \{\hat{\mu} : \mu \in B(\Gamma), \hat{\mu} \text{ vanishes at infinity}\}$. $B_0(\Gamma) \cup \{\delta_0\}$ is a closed subalgebra of $B(\Gamma)$ which satisfies (i)–(iii). Thus our Theorem 1 yields a variant of Varopoulos' result.

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