

**INTEGRAL REPRESENTATION OF MULTIPLICATIVE,
INVOLUTION PRESERVING OPERATORS IN
 $\mathfrak{L}(C(S), E)^1$**

GERALD W. JOHNSON

1. Introduction. Throughout, S will be a compact Hausdorff space and E will be a Banach space which is the dual of another Banach space F . $C(S)$ will denote the space of complex-valued continuous functions on S topologized with the topology of uniform convergence. $\mathfrak{L}(C(S), E)$ will denote the space of continuous linear operators from $C(S)$ to E . A theorem of Gil de Lamadrid [5, p. 103] identifies $\mathfrak{L}(C(S), E)$ with a space of E -valued "measures," the correspondence between operator and measure being given by integration. A closely related result was given earlier by Bartle, Dunford, and Schwartz [2]. Now if E is a Banach algebra with involution [7, p. 178], it makes sense to consider operators which are not only continuous and linear but which also preserve multiplication and involution. A natural question arises: How are these additional properties reflected in the representing measure? We answer this question under additional restrictions on E . We also give several examples of spaces E satisfying the hypotheses of our theorems. One can use the results of this paper to prove the Spectral Theorem for bounded operators; but the proof follows standard lines and will not be included (see [6, p. 99]).

We conclude this introduction with a precise description of Gil de Lamadrid's Theorem. Our description differs from Gil de Lamadrid's, but it is not difficult to verify that they are equivalent. We consider the class $N(S, E)$ of set functions m from $\mathfrak{B}(S)$, the Borel class of S , to E which are countably additive and regular with respect to the weak topology $\sigma(E, F)$ on E induced by F . To say that $m: \mathfrak{B}(S) \rightarrow E$ is regular with respect to the topology $\sigma(E, F)$ means that, for every $B \in \mathfrak{B}(S)$ and every $\sigma(E, F)$ -neighborhood N of 0 , there exists a compact set K and an open set U such that $K \subseteq B \subseteq U$ and, if $A \subseteq U - K$, then $m(A) \in N$. Defining addition and scalar-multiplication in $N(S, E)$ in the usual set-wise fashion; i.e., $(m_1 + m_2)(B) = m_1(B) + m_2(B)$ and $(\alpha m_1)(B) = \alpha m_1(B)$, $N(S, E)$ is a vector space. The following formula defines a norm on $N(S, E)$ making it a Banach space:

Received by the editors December 7, 1968.

¹ This paper is based on Chapter 5 of the author's Ph.D. Thesis [6] written at the University of Minnesota under the direction of Professor Gil de Lamadrid. The work was supported in part by the National Science Foundation.

$\|m\| = \sup \left\{ \left\| \sum_{j=1}^n \alpha_j m(B_j) \right\| \right\}$ where B_1, \dots, B_n is a partition of S by elements of $\mathfrak{B}(S)$ and each $\alpha_j \in C$, the field of complex numbers, satisfying $|\alpha_j| \leq 1$. Now given $m \in N(S, E)$ we define $L: C(S) \rightarrow E$ by the formula: $L(f) = \int_S f dm$ where the integration may be interpreted via [1]. Gil de Lamadrid's Theorem says that this correspondence sets up an isometric isomorphism of $N(S, E)$ onto $\mathfrak{L}(C(S), E)$.

2. The representation theorems. Our first theorem deals with multiplicative operators. The proof is similar in outline to the proof of the Spectral Theorem as in [4, circa p. 897].

THEOREM 1. *Let E be a Banach algebra which, as a Banach space, is the dual of another Banach space F ; further suppose that the multiplication in E is separately continuous with respect to the topology $\sigma(E, F)$. Let $L \in \mathfrak{L}(C(S), E)$ and let $m \in N(S, E)$ be its representing measure. L satisfies $L(fg) = L(f)L(g)$ for all $f, g \in C(S)$ if and only if $m(B_1 \cap B_2) = m(B_1)m(B_2)$ for all $B_1, B_2 \in \mathfrak{B}(S)$.*

PROOF. We begin by giving some useful facts in integration theory. Let $m \in N(S, E)$ and let $e \in E$. We define $m_e: \mathfrak{B}(S) \rightarrow E$ by $m_e(B) = m(B)e$, and we let $m^e: \mathfrak{B}(S) \rightarrow E$ be defined by $m^e(B) = em(B)$. Since multiplication in E is separately continuous with respect to the topology $\sigma(E, F)$, we easily see that m_e and m^e are in $N(S, E)$. For $f \in C(S)$, we get $\int_S f dm_e = (\int_S f dm)e$ and $\int_S f dm^e = e(\int_S f dm)$. These formulas are clear for f a simple function and are obtained for $f \in C(S)$ by passage to the limit (see [1, p. 341]).

Now let $B_0 \in \mathfrak{B}(S)$ and define $m_{B_0}: \mathfrak{B}(S) \rightarrow E$ by $m_{B_0}(B) = m(B \cap B_0)$. $m_{B_0} \in N(S, E)$ and we get $\int_S f dm_{B_0} = \int_{B_0} f dm$, $f \in C(S)$.

We need one additional fact from integration theory. If $g \in C(S)$, we define $m_g: \mathfrak{B}(S) \rightarrow E$ by $m_g(B) = \int_B g dm$. We claim that $m_g \in N(S, E)$ and that, for $f \in C(S)$, $\int_S f dm_g = \int_S f g dm$.

Let $M(S)$ be the set of countably additive, complex-valued, regular Borel measures on S . $N(S, E)$ may alternately be described as the class of set functions $m: \mathfrak{B}(S) \rightarrow E$ such that $m(\cdot)(x) \in M(S)$ for all $x \in F$. Let $x \in F$. It is not difficult to establish that $m_g(B)(x) = \int_B g dm(\cdot)(x)$. Hence $m_g(\cdot)(x)$ is the indefinite integral of $m(\cdot)(x) \in M(S)$. Such an indefinite integral is countably additive [3, p. 152]. $m_g(\cdot)(x)$ must be regular since $m(\cdot)(x)$ is regular. This follows from [3, p. 114]. Thus $m_g \in N(S, E)$.

To establish the formula, it suffices to show that $(\int_S f dm_g)(x) = (\int_S f g dm)(x)$ for all $x \in F$; that is, it suffices to show $\int_S f dm_g(\cdot)(x) = \int_S f g dm(\cdot)(x)$ for $x \in F$. But this formula is given for complex measures in [3, p. 180].

Now suppose $L(fg) = L(f)L(g)$ for all $f, g \in C(S)$ and let $B_1, B_2 \in \mathfrak{B}(S)$. We wish to show that $m(B_1 \cap B_2) = m(B_1)m(B_2)$. It suffices to show that $m_{B_2} = m_{m(B_2)}$. Let $g \in C(S)$. Since the Gil de Lamadrid Theorem gives a one-to-one correspondence between $\mathfrak{L}(C(S), E)$ and $N(S, E)$, it suffices to show that $\int_S g dm_{B_2} = \int_S g dm_{m(B_2)}$. Now $\int_S g dm_{B_2} = \int_{B_2} g dm = m_g(B_2)$. Also $\int_S g dm_{m(B_2)} = (\int_S g dm)m(B_2) = L(g)m(B_2)$. Thus to show $\int_S g dm_{B_2} = \int_S g dm_{m(B_2)}$, it suffices to show that $m_g = m^{L(g)}$. Let $f \in C(S)$. It suffices to show that $\int_S f dm_g = \int_S f dm^{L(g)}$. However, $\int_S f dm_g = \int_S f g dm = L(fg) = L(g)L(f) = L(g)\int_S f dm = \int_S f dm^{L(g)}$. $m(B_1 \cap B_2) = m(B_1)m(B_2)$ now follows.

The converse is easier to establish; we outline the argument. Suppose $m \in N(S, E)$ satisfying $m(B_1 \cap B_2) = m(B_1)m(B_2)$ for all $B_1, B_2 \in \mathfrak{B}(S)$ and let L be the corresponding operator. Let $f, g \in C(S)$. We must establish the formula $\int_S f g dm = (\int_S f dm)(\int_S g dm)$. Let $f = \chi_B$ be the characteristic function of $B \in \mathfrak{B}(S)$. When g is also the characteristic function of a set in $\mathfrak{B}(S)$, the formula is obvious. The formula follows from the linearity of the integral [1, p. 342] when g is a simple function. For $g \in C(S)$, the formula follows by passage to the limit. We now have the formula for $f = \chi_B$ and $g \in C(S)$. The case where f is a simple function follows by linearity. Finally, by using Bartle's Bounded Convergence Theorem [1, p. 345] and passing to the limit, we get the result for arbitrary $f \in C(S)$. This completes the proof.

Our second theorem deals with involution preserving members of $\mathfrak{L}(C(S), E)$. If $f \in C(S)$, f^* will denote the complex conjugate of f . The map $f \rightarrow f^*$ is an involution on $C(S)$ and, in fact, $C(S)$ is a Banach algebra with isometric involution.

THEOREM 2. *Let E be a Banach algebra with isometric involution*. Suppose that E , as a Banach space, is the dual of another Banach space F . Finally, suppose that the involution is continuous with respect to the topology $\sigma(E, F)$. Let $L \in \mathfrak{L}(C(S), E)$ and let m be its representing measure. L satisfies $L(f^*) = L(f)^*$ for all $f \in C(S)$ if and only if $m(B) = m(B)^*$ for all $B \in \mathfrak{B}(S)$.*

PROOF. Let $m \in N(S, E)$. We claim that the set function $m^*: \mathfrak{B}(S) \rightarrow E$ defined by $m^*(B) = m(B)^*$ is in $N(S, E)$. Let $x \in F$. It suffices to show that $m^*(\cdot)(x) \in M(S)$. Define $x_0: E \rightarrow C$ by the equation $x_0(e) = x(e^*)^-$ where the bar denotes the conjugate of the complex number $x(e^*)$. x_0 is linear and, because of the assumption that $*$ is continuous with respect to the topology $\sigma(E, F)$, and because the map $e \rightarrow x(e^*)^-$ is composed of the maps $e \rightarrow e^* \rightarrow x(e^*) \rightarrow x(e^*)^-$, we see that x_0 is a continuous linear functional on $(E, \sigma(E, F))$. Hence x_0 may be regarded as an element of F [9, p. 124]. Now $m(B)(x_0) = m(B)^*(x)^--$

$= m^*(B)(x)$. But $m(\cdot)(x_0) \in M(S)$ and so $m(\cdot)(x_0)^- \in M(S)$ and hence $m^*(\cdot)(x) \in M(S)$.

Next we observe that the formula $(\int_S f dm)^* = \int_S f^* dm^*$ is valid for $f \in C(S)$ and $m \in N(S, E)$. This is established in the usual way.

Now suppose $L(f^*) = L(f)^*$ for all $f \in C(S)$ and let m be the representing measure for L . Now $\int_S f^* dm = (\int_S f dm)^* = \int_S f^* dm^*$. Since this holds for all $f \in C(S)$ and since $m, m^* \in N(S, E)$, Gil de Lamadrid's Theorem implies that $m = m^*$.

Conversely, suppose $m = m^*$; then $(\int_S f dm)^* = \int_S f^* dm^* = \int_S f^* dm$. Thus $L(f)^* = L(f^*)$. This completes the proof.

Next we list some Banach algebras to which Theorems 1 and 2 may be applied: (a) If X is a reflexive Banach space, $\mathfrak{L}(X)$ satisfies the hypotheses of Theorem 1. We mention that $\mathfrak{L}(X)$ is the dual of $X \otimes_\gamma X'$ [11, p. 47]. (b) Both Theorems 1 and 2 are applicable to any B^* -algebra [9, p. 180] which, as a Banach space, is the dual of another Banach space. These are exactly the B^* -algebras identifiable with W^* -algebras ([8] and [12]). $\mathfrak{L}(H)$ is such a B^* -algebra where H is a Hilbert space. If (S, Σ, μ) is a measure space such that $L'_1(S, \Sigma, \mu) = L_\infty(S, \Sigma, \mu)$ (see [3, p. 289]), then $L_\infty(S, \Sigma, \mu)$ is another such B^* -algebra. (c) Theorems 1 and 2 apply to any H^* -algebra [7, p. 272]. In particular they apply to $L_2(G)$ for any compact group G where convolution is taken as multiplication [7, p. 330]. Also they apply to the Schmidt class of operators on a Hilbert space ([7, pp. 285–288] and [10, pp. 29–36]). (d) Let G be a locally compact topological group. With convolution as multiplication, $M(G)$ satisfies the hypotheses of both theorems. (e) The trace class of operators on a Hilbert space [10, p. 37] also satisfies the hypotheses of both theorems.

BIBLIOGRAPHY

1. R. G. Bartle, *A general bilinear vector integral*, *Studia Math.* 15 (1956), 337–352.
2. R. G. Bartle, N. Dunford and J. Schwartz, *Weak compactness and vector measures*, *Canad. J. Math.* 7 (1955), 289–305.
3. N. Dunford and J. Schwartz, *Linear operators. Part I: General theory*, Interscience, New York and London, 1958.
4. ———, *Linear operators. Part II: Spectral theory self-adjoint operators in hilbert space*, Interscience, New York and London, 1964.
5. J. Gil de Lamadrid, *Measures and tensors*, *Trans. Amer. Math. Soc.* 114 (1965), 98–121.
6. Gerald W. Johnson, *Integral representations of certain linear operators*, Thesis, University of Minnesota, Minneapolis, Minn., 1968.
7. C. E. Rickart, *General theory of banach algebras*, Van Nostrand, Princeton, N. J., 1960.
8. S. Sakai, *A characterization of W^* -algebras*, *Pacific J. Math.* 6(1956), 763–773.

9. H. H. Schaefer, *Topological vector spaces*, Macmillan, New York, 1966.
10. R. Schatten, *Norm ideals of completely continuous operators*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Heft 27, Springer-Verlag, Berlin, 1960.
11. ———, *A theory of cross-spaces*, Ann. of Math. Studies, no. 26, Princeton Univ. Press, Princeton, N. J., 1950.
12. P. C. Shields, *A new topology for von Neumann algebras*, Bull. Amer. Math. Soc. **65** (1959), 267–269.

UNIVERSITY OF NEBRASKA