INTEGRAL REPRESENTATION OF MULTIPLICATIVE, INVOLUTION PRESERVING OPERATORS IN $\mathcal{L}(C(S), E)$[1]

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1. **Introduction.** Throughout, S will be a compact Hausdorff space and E will be a Banach space which is the dual of another Banach space F. $C(S)$ will denote the space of complex-valued continuous functions on S topologized with the topology of uniform convergence. $\mathcal{L}(C(S), E)$ will denote the space of continuous linear operators from $C(S)$ to E. A theorem of Gil de Lamadrid [**5**, p. 103] identifies $\mathcal{L}(C(S), E)$ with a space of E-valued "measures," the correspondence between operator and measure being given by integration. A closely related result was given earlier by Bartle, Dunford, and Schwartz [**2**]. Now if E is a Banach algebra with involution [**7**, p. 178], it makes sense to consider operators which are not only continuous and linear but which also preserve multiplication and involution. A natural question arises: How are these additional properties reflected in the representing measure? We answer this question under additional restrictions on E. We also give several examples of spaces E satisfying the hypotheses of our theorems. One can use the results of this paper to prove the Spectral Theorem for bounded operators; but the proof follows standard lines and will not be included (see [**6**, p. 99]).

We conclude this introduction with a precise description of Gil de Lamadrid's Theorem. Our description differs from Gil de Lamadrid's, but it is not difficult to verify that they are equivalent. We consider the class $N(S, E)$ of set functions m from $\mathcal{B}(S)$, the Borel class of S, to E which are countably additive and regular with respect to the weak topology $\sigma(E, F)$ on E induced by F. To say that $m: \mathcal{B}(S)\to E$ is regular with respect to the topology $\sigma(E, F)$ means that, for every $B\in\mathcal{B}(S)$ and every $\sigma(E, F)$-neighborhood N of 0, there exists a compact set K and an open set U such that $K\subseteq B\subseteq U$ and, if $A\subseteq U-K$, then $m(A)\in N$. Defining addition and scalar-multiplication in $N(S, E)$ in the usual set-wise fashion; i.e., $(m_1+m_2)(B)=m_1(B)+m_2(B)$ and $(\alpha m_1)(B)=\alpha m_1(B)$, $N(S, E)$ is a vector space. The following formula defines a norm on $N(S, E)$ making it a Banach space:

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$\|m\| = \sup\{\|\sum_{j=1}^{n} \alpha_j m(B_j)\|\}$ where $B_1, \cdots, B_n$ is a partition of S by elements of $\mathcal{B}(S)$ and each $\alpha_j \in C$, the field of complex numbers, satisfying $|\alpha_j| \leqq 1$. Now given $m \in N(S, E)$ we define L: $C(S) \to E$ by the formula: $L(f) = \int_S f dm$ where the integration may be interpreted via [1]. Gil de Lamadrid's Theorem says that this correspondence sets up an isometric isomorphism of $N(S, E)$ onto $\mathcal{L}(C(S), E)$.

2. **The representation theorems.** Our first theorem deals with multiplicative operators. The proof is similar in outline to the proof of the Spectral Theorem as in [**4**, circa p. 897].

THEOREM 1. *Let E be a Banach algebra which, as a Banach space, is the dual of another Banach space F; further suppose that the multiplication in E is separately continuous with respect to the topology $\sigma(E, F)$. Let $L \in \mathcal{L}(C(S), E)$ and let $m \in N(S, E)$ be its representing measure. L satisfies $L(fg) = L(f)L(g)$ for all $f, g \in C(S)$ if and only if $m(B_1 \cap B_2) = m(B_1)m(B_2)$ for all $B_1, B_2 \in \mathcal{B}(S)$.*

PROOF. We begin by giving some useful facts in integration theory. Let $m \in N(S, E)$ and let $e \in E$. We define m_e: $\mathcal{B}(S) \to E$ by $m_e(B) = m(B)e$, and we let m^e: $\mathcal{B}(S) \to E$ be defined by $m^e(B) = em(B)$. Since multiplication in E is separately continuous with respect to the topology $\sigma(E, F)$, we easily see that m_e and m^e are in $N(S, E)$. For $f \in C(S)$, we get $\int_S f dm_e = (\int_S f dm)e$ and $\int_S f dm^e = e(\int_S f dm)$. These formulas are clear for f a simple function and are obtained for $f \in C(S)$ by passage to the limit (see [**1**, p. 341]).

Now let $B_0 \in \mathcal{B}(S)$ and define m_{B_0}: $\mathcal{B}(S) \to E$ by $m_{B_0}(B) = m(B \cap B_0)$. $m_{B_0} \in N(S, E)$ and we get $\int_S f dm_{B_0} = \int_{B_0} f dm$, $f \in C(S)$.

We need one additional fact from integration theory. If $g \in C(S)$, we define m_g: $\mathcal{B}(S) \to E$ by $m_g(B) = \int_B g dm$. We claim that $m_g \in N(S, E)$ and that, for $f \in C(S)$, $\int_S f dm_g = \int_S fg dm$.

Let $M(S)$ be the set of countably additive, complex-valued, regular Borel measures on S. $N(S, E)$ may alternately be described as the class of set functions m: $\mathcal{B}(S) \to E$ such that $m(\cdot)(x) \in M(S)$ for all $x \in F$. Let $x \in F$. It is not difficult to establish that $m_g(B)(x) = \int_B g dm(\cdot)(x)$. Hence $m_g(\cdot)(x)$ is the indefinite integral of $m(\cdot)(x) \in M(S)$. Such an indefinite integral is countably additive [**3**, p. 152]. $m_g(\cdot)(x)$ must be regular since $m(\cdot)(x)$ is regular. This follows from [**3**, p. 114]. Thus $m_g \in N(S, E)$.

To establish the formula, it suffices to show that $(\int_S f dm_g)(x) = (\int_S fg dm)(x)$ for all $x \in F$; that is, it suffices to show $\int_S f dm_g(\cdot)(x) = \int_S fg dm(\cdot)(x)$ for $x \in F$. But this formula is given for complex measures in [**3**, p. 180].

Now suppose $L(fg)=L(f)L(g)$ for all f, $g\in C(S)$ and let B_1, $B_2 \in \mathcal{B}(S)$. We wish to show that $m(B_1\cap B_2)=m(B_1)m(B_2)$. It suffices to show that $m_{B_2}=m_{m(B_2)}$. Let $g\in C(S)$. Since the Gil de Lamadrid Theorem gives a one-to-one correspondence between $\mathfrak{L}(C(S), E)$ and $N(S, E)$, it suffices to show that $\int_S g\,dm_{B_2}=\int_S g\,dm_{m(B_2)}$. Now $\int_S g\,dm_{B_2} =\int_{B_2} g\,dm=m_g(B_2)$. Also $\int_S g\,dm_{m(B_2)}=(\int_S g\,dm)m(B_2)=L(g)m(B_2)$. Thus to show $\int_S g\,dm_{B_2}=\int_S g\,dm_{m(B_2)}$, it suffices to show that $m_g=m^{L(g)}$. Let $f\in C(S)$. It suffices to show that $\int_S f\,dm_g=\int_S f\,dm^{L(g)}$. However, $\int_S f\,dm_g = \int_S fg\,dm = L(fg) = L(g)L(f) = L(g)\int_S f\,dm =\int_S f\,dm^{L(g)}$. $m(B_1\cap B_2)=m(B_1)m(B_2)$ now follows.

The converse is easier to establish; we outline the argument. Suppose $m\in N(S, E)$ satisfying $m(B_1\cap B_2)=m(B_1)m(B_2)$ for all B_1, $B_2 \in\mathcal{B}(S)$ and let L be the corresponding operator. Let f, $g\in C(S)$. We must establish the formula $\int_S fg\,dm=(\int_S f\,dm)(\int_S g\,dm)$. Let $f=\chi_B$ be the characteristic function of $B\in\mathcal{B}(S)$. When g is also the characteristic function of a set in $\mathcal{B}(S)$, the formula is obvious. The formula follows from the linearity of the integral [**1**, p. 342] when g is a simple function. For $g\in C(S)$, the formula follows by passage to the limit. We now have the formula for $f=\chi_B$ and $g\in C(S)$. The case where f is a simple function follows by linearity. Finally, by using Bartle's Bounded Convergence Theorem [**1**, p. 345] and passing to the limit, we get the result for arbitrary $f\in C(S)$. This completes the proof.

Our second theorem deals with involution preserving members of $\mathfrak{L}(C(S), E)$. If $f\in C(S)$, f^* will denote the complex conjugate of f. The map $f\rightarrow f^*$ is an involution on $C(S)$ and, in fact, $C(S)$ is a Banach algebra with isometric involution.

THEOREM 2. *Let E be a Banach algebra with isometric involution*. Suppose that E, as a Banach space, is the dual of another Banach space F. Finally, suppose that the involution is continuous with respect to the topology $\sigma(E, F)$. Let $L\in\mathfrak{L}(C(S), E)$ and let m be its representing measure. L satisfies $L(f^*)=L(f)^*$ for all $f\in C(S)$ if and only if $m(B) =m(B)^*$ for all $B\in\mathcal{B}(S)$.*

PROOF. Let $m\in N(S, E)$. We claim that the set function $m^*:\mathcal{B}(S)\rightarrow E$ defined by $m^*(B)=m(B)^*$ is in $N(S, E)$. Let $x\in F$. It suffices to show that $m^*(\cdot)(x)\in M(S)$. Define x_0: $E\rightarrow C$ by the equation $x_0(e) =x(e^*)^-$ where the bar denotes the conjugate of the complex number $x(e^*)$. x_0 is linear and, because of the assumption that * is continuous with respect to the topology $\sigma(E, F)$, and because the map $e\rightarrow x(e^*)^-$ is composed of the maps $e\rightarrow e^*\rightarrow x(e^*)\rightarrow x(e^*)^-$, we see that x_0 is a continuous linear functional on $(E, \sigma(E, F))$. Hence x_0 may be regarded as an element of F [**9**, p. 124]. Now $m(B)(x_0)^-=m(B)^*(x)^{--}$

$=m^*(B)(x)$. But $m(\cdot)(x_0)\in M(S)$ and so $m(\cdot)(x_0)^-\in M(S)$ and hence $m^*(\cdot)(x)\in M(S)$.

Next we observe that the formula $(\int_S fdm)^*=\int_S f^*dm^*$ is valid for $f\in C(S)$ and $m\in N(S, E)$. This is established in the usual way.

Now suppose $L(f^*)=L(f)^*$ for all $f\in C(S)$ and let m be the representing measure for L. Now $\int_S f^*dm=(\int_S fdm)^*=\int_S f^*dm^*$. Since this holds for all $f\in C(S)$ and since $m, m^*\in N(S, E)$, Gil de Lamadrid's Theorem implies that $m=m^*$.

Conversely, suppose $m=m^*$; then $(\int_S fdm)^*=\int_S f^*dm^*=\int_S f^*dm$. Thus $L(f)^*=L(f^*)$. This completes the proof.

Next we list some Banach algebras to which Theorems 1 and 2 may be applied: (a) If X is a reflexive Banach space, $\mathfrak{L}(X)$ satisfies the hypotheses of Theorem 1. We mention that $\mathfrak{L}(X)$ is the dual of $X\otimes_\gamma X'$ [**11**, p. 47]. (b) Both Theorems 1 and 2 are applicable to any B^*-algebra [**9**, p. 180] which, as a Banach space, is the dual of another Banach space. These are exactly the B^*-algebras identifiable with W^*-algebras ([**8**] and [**12**]). $\mathfrak{L}(H)$ is such a B^*-algebra where H is a Hilbert space If (S, Σ, μ) is a measure space such that $L_1'(S, \Sigma, \mu)$ $=L_\infty(S, \Sigma, \mu)$ (see [**3**, p. 289]), then $L_\infty(S, \Sigma, \mu)$ is another such B^*-algebra. (c) Theorems 1 and 2 apply to any H^*-algebra [**7**, p. 272]. In particular they apply to $L_2(G)$ for any compact group G where convolution is taken as multiplication [**7**, p. **330**]. Also they apply to the Schmidt class of operators on a Hilbert space ([**7**, pp. 285–288] and [**10**, pp. 29–36]). (d) Let G be a locally compact topological group. With convolution as multiplication, $M(G)$ satisfies the hypotheses of both theorems. (e) The trace class of operators on a Hilbert space [**10**, p. 37] also satisfies the hypotheses of both theorems.

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