

SYMMETRIC OPERATORS WITH TWICE CONTINUOUSLY DIFFERENTIABLE SPECTRAL FUNCTIONS

RICHARD C. GILBERT¹

In a previous article [2] it was shown that if A is a simple closed symmetric operator with deficiency indices $(1, 1)$ in a Hilbert space H and if A has a selfadjoint extension A_0 in H with a discrete spectrum, then every selfadjoint extension and every minimal selfadjoint dilation A^+ of A has spectral multiplicity 1. Further, it was possible to make certain statements about the spectrum of A^+ . In the present article it is shown that if A has a selfadjoint extension A_0 with a twice continuously differentiable spectral function, whose derivative is always positive, then there are minimal selfadjoint dilations A^+ of A for which the spectral multiplicity is two; indeed, there are minimal selfadjoint dilations A^+ which are unitarily equivalent to the multiplication operator in $L^2(-\infty, \infty) \oplus L^2(-\infty, \infty)$. As in [2] the proof of these facts depends upon obtaining an expansion theorem which is valid for each of the selfadjoint extensions or dilations A^+ . This expansion theorem is the analog of the well-known expansion theorems for linear ordinary differential operators; however, it is necessary in deriving it to overcome the fact that, unlike a linear ordinary differential operator, there does not necessarily exist a basis of entire functions for the solutions of the equation $Af = \lambda f$. After the expansion theorem is obtained, it is used to define a unitary map of H^+ (the space in which A^+ acts) onto $L^2(-\infty, \infty) \oplus L^2(-\infty, \infty)$. The same map takes A^+ into the multiplication operator in $L^2(-\infty, \infty) \oplus L^2(-\infty, \infty)$.

Throughout this article we shall adhere to the notation of [2], except that we shall use the symbol $Q(\lambda)$ in place of $Q_1(\lambda)$.

We begin with a lemma on the limit of a function analytic in the upper half-plane as one approaches the real axis.

LEMMA. *Suppose for all nonreal λ , $\theta(\lambda) = \int_{-\infty}^{\infty} (t-\lambda)^{-1} d\rho(t)$, where $\rho(t)$ is a nondecreasing bounded function which is twice continuously differentiable for all t . Then, $\theta(\lambda)$ may be defined continuously down to the*

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real axis. For a point ξ on the real axis we shall have

$$\operatorname{Re} \theta(\xi) = P \int_{-\infty}^{\infty} (t - \xi)^{-1} d\rho(t),$$

$$\operatorname{Im} \theta(\xi) = \pi \rho'(\xi),$$

where P stands for Cauchy principal value.

PROOF. Let $f(\xi, t) = [\rho'(t) - \rho'(\xi)](t - \xi)^{-1}$ if $t \neq \xi$, and let $f(\xi, t) = \rho''(t)$ if $t = \xi$. Then, $f(\xi, t)$ is a continuous function, and $\rho'(t) = \rho'(\xi) + (t - \xi)f(\xi, t)$.

Now,

$$\begin{aligned} P \int_{-\infty}^{\infty} (t - \xi)^{-1} d\rho(t) &= \int_{-\infty}^{\xi-1} (t - \xi)^{-1} d\rho(t) + \int_{\xi+1}^{\infty} (t - \xi)^{-1} d\rho(t) + \int_{\xi-1}^{\xi+1} f(\xi, t) dt \\ &= \int_{-\infty}^{-1} u^{-1} \rho'(u + \xi) du + \int_1^{\infty} u^{-1} \rho'(u + \xi) du + \int_{-1}^1 f(\xi, u + \xi) du. \end{aligned}$$

From this last expression it is not difficult to see that $P \int_{-\infty}^{\infty} (t - \xi)^{-1} d\rho(t)$ exists and is a continuous function of ξ . If $\lambda = \xi + i\eta$, then

$$\begin{aligned} \operatorname{Re} \theta(\lambda) &= \int_{-\infty}^{\infty} (t - \xi) [(t - \xi)^2 + \eta^2]^{-1} d\rho(t) \\ &= \int_{-\infty}^{-1} [u + (\xi_0 - \xi)] \{ [u + (\xi_0 - \xi)]^2 + \eta^2 \}^{-1} \rho'(u + \xi_0) du \\ &\quad + \int_1^{\infty} [u + (\xi_0 - \xi)] \{ [u + (\xi_0 - \xi)]^2 + \eta^2 \}^{-1} \rho'(u + \xi_0) du \\ &\quad + (1/2) \rho'(\xi) \log \{ [1 - (\xi - \xi_0)]^2 + \eta^2 \} \{ [1 + (\xi - \xi_0)]^2 + \eta^2 \}^{-1} \\ &\quad + \int_{-1}^1 f(\xi, u + \xi_0) du - \eta^2 \int_{-1}^1 \{ [u - (\xi - \xi_0)]^2 + \eta^2 \}^{-1} \\ &\quad \cdot f(\xi, u + \xi_0) du. \end{aligned}$$

From this last expression it is not difficult to see that $\lim_{\lambda \rightarrow \xi_0} \operatorname{Re} \theta(\lambda) = P \int_{-\infty}^{\infty} (t - \xi_0)^{-1} d\rho(t)$. Hence, $\operatorname{Re} \theta(\lambda)$ can be defined continuously down to the real axis, and its value on the real axis is $P \int_{-\infty}^{\infty} (t - \xi)^{-1} d\rho(t)$.

Now,

$$\begin{aligned}
 \operatorname{Im} \theta(\lambda) &= \eta \int_{-\infty}^{\infty} [(t - \xi)^2 + \eta^2]^{-1} d\rho(t) \\
 &= \eta \int_{-\infty}^{-1} \{ [u - (\xi - \xi_0)]^2 + \eta^2 \}^{-1} \rho'(u + \xi_0) du \\
 &\quad + \eta \int_1^{\infty} \{ [u - (\xi - \xi_0)]^2 + \eta^2 \}^{-1} \rho'(u + \xi_0) du \\
 &\quad + \rho'(\xi) \{ \tan^{-1} \eta^{-1} [1 - (\xi - \xi_0)] + \tan^{-1} \eta^{-1} [1 + (\xi - \xi_0)] \} \\
 &\quad + \eta \int_{-1}^1 [u - (\xi - \xi_0)] \{ [u - (\xi - \xi_0)]^2 + \eta^2 \}^{-1} f(\xi, u + \xi_0) du.
 \end{aligned}$$

From this expression it is not difficult to see that $\lim_{\lambda \rightarrow \xi_0} \operatorname{Im} \theta(\lambda) = \pi \rho'(\xi_0)$. Since $\rho'(\xi)$ is continuous, we see that $\operatorname{Im} \theta(\lambda)$ can be defined continuously down to the real axis, and its value on the real axis is $\pi \rho'(\xi)$.

This completes the proof of the lemma.

Let A be a simple closed symmetric operator with deficiency indices $(1, 1)$ in the Hilbert space H . Let λ_0 be a complex number with positive imaginary part, and let g_0 be an element of norm 1 in the deficiency subspace of A , corresponding to λ_0^- . (Here λ_0^- stands for complex conjugate.) Suppose that A_0 is a selfadjoint extension of A in H . Since A is simple, g_0 is a generating element for A_0 . (See Achieser and Glasmann [1, §81, Theorem 2].) Hence, A_0 is unitarily equivalent to the multiplication operator in $L^2_{\rho_0}$, where $\rho_0(t) = (E_0(t)g_0, g_0)$ and $E_0(t)$ is the spectral function of A_0 . By S we shall denote those elements f in H whose transforms $F(t)$ in $L^2_{\rho_0}$ are continuously differentiable functions. S is dense in H .

Suppose that $R_0(\lambda)$ is the resolvent of A_0 . Let $g(\lambda) = g_0 + (\lambda - \lambda_0) \cdot R_0(\lambda)g_0$, and let $Q(\lambda) = i \operatorname{Im} \lambda_0 + (\lambda - \lambda_0)(g_0, g(\lambda^-))$. As is indicated in [2, Lemma 2], for $\operatorname{Im} \lambda \neq 0$ the generalized resolvent $R(\lambda)$ of A corresponding to a selfadjoint extension or dilation A^+ of A has the form

$$(1) \quad R(\lambda) = R_0(\lambda) - [\theta(\lambda) + Q(\lambda)]^{-1}(\cdot, g(\lambda^-))g(\lambda),$$

where $\theta(\lambda)$ is an analytic function for $\operatorname{Im} \lambda \neq 0$ which has nonnegative imaginary part in the upper half-plane. $R(\lambda)$ corresponds to a selfadjoint extension in the space H if and only if $\theta(\lambda)$ is identically equal to a constant θ , $-\infty < \theta \leq +\infty$.

THEOREM 1 (EXPANSION THEOREM). *Suppose $\rho_0(t)$ is twice continuously differentiable and that $\rho'_0(t) > 0$ for all t . Let A^+ be a selfadjoint*

extension of A in H or a selfadjoint dilation of A in $H^+ \supset H$. Let $E^+(\lambda)$ be the spectral function of A^+ . Suppose the function $\theta(\lambda)$ of equation (1) has the form $\theta(\lambda) = a + b\lambda + \int_{-\infty}^{\infty} (t - \lambda)^{-1} d\rho(t)$, where $b \geq 0$, a is real, $\rho(t)$ is nondecreasing, bounded and twice continuously differentiable. Then for any numbers $\alpha, \beta, \alpha < \beta$, and for any two elements $f, h \in S$,

$$(2) \quad ([E^+(\beta) - E^+(\alpha)]f, h) = \int_{\alpha}^{\beta} \sum_{i,j=1}^2 C_i(f; \xi) [C_j(h; \xi)]^{-\delta_{ij}(\xi)} d\xi,$$

where

$$C_1(f; \xi) = (f, g_0) + (\xi - \lambda_0^-)P \int_{-\infty}^{\infty} (t - \xi)^{-1} d(f, E_0(t)g_0),$$

$$C_2(f; \xi) = \pi(\xi - \lambda_0^-)(f, E_0(\xi)g_0)', \quad \text{where the prime stands for differentiation,}$$

$$\delta_{11}(\xi) = B(\xi)/\pi[A^2(\xi) + B^2(\xi)],$$

$$\delta_{12}(\xi) = \delta_{21}(\xi) = -A(\xi)/\pi[A^2(\xi) + B^2(\xi)],$$

$$\delta_{22}(\xi) = 1/\pi^2 \left| \xi - \lambda_0 \right|^2 \rho_0'(\xi) - B(\xi)/\pi[A^2(\xi) + B^2(\xi)],$$

$$A(\xi) = P \int_{-\infty}^{\infty} (t - \xi)^{-1} d\rho(t) + a + b\xi + (\xi - \operatorname{Re} \lambda_0)$$

$$+ \left| \xi - \lambda_0 \right|^2 P \int_{-\infty}^{\infty} (t - \xi)^{-1} d\rho_0(t),$$

$$B(\xi) = \pi[\rho'(\xi) + \left| \xi - \lambda_0 \right|^2 \rho_0'(\xi)].$$

PROOF. By the process of polarization applied to $(E_0(t)f, h)$, we see that

$$(3) \quad \begin{aligned} (R_0(\lambda)f, h) &= \int_{-\infty}^{\infty} (t - \lambda)^{-1} d(E_0(t)f, h) \\ &= (1/4) \int_{-\infty}^{\infty} (t - \lambda)^{-1} d(E_0(t)(f + h), f + h) \\ &\quad - (1/4) \int_{-\infty}^{\infty} (t - \lambda)^{-1} d(E_0(t)(f - h), f - h) \\ &\quad + (i/4) \int_{-\infty}^{\infty} (t - \lambda)^{-1} d(E_0(t)(f + ih), f + ih) \\ &\quad - (i/4) \int_{-\infty}^{\infty} (t - \lambda)^{-1} d(E_0(t)(f - ih), f - ih). \end{aligned}$$

Since $(E_0(t)(f+h), f+h) = \int_{-\infty}^t |F(\xi) + H(\xi)|^2 d\rho_0(\xi)$, and since $F(\xi)$, $H(\xi)$ are continuously differentiable, we see that $(E_0(t)(f+h), f+h) \in C^2$. The same is true for the other terms in (3). It follows from the lemma, then, that $(R_0(\lambda)f, h)$ can be defined continuously down to the real axis and that at a point ξ on the real axis,

$$(4) \quad (R_0(\xi)f, h) = P \int_{-\infty}^{\infty} (t - \xi)^{-1} d(E_0(t)f, h) + i\pi(E_0(\xi)f, h)'.$$

Since $(E_0(t)f, h) = \int_{-\infty}^t F(u)[H(u)]^{-\rho'_0(u)} du$, equation (4) can be written

$$(5) \quad (R_0(\xi)f, h) = P \int_{-\infty}^{\infty} (t - \xi)^{-1} F(t)[H(t)]^{-\rho'_0(t)} dt \\ + i\pi F(\xi)[H(\xi)]^{-\rho'_0(\xi)}.$$

One can show similarly that $(f, g(\lambda^-))$ and $(g(\lambda), h)$ can be defined continuously down to the real axis and that at a point ξ on the real axis,

$$(6) \quad (f, g(\xi)) = C_1(f; \xi) + iC_2(f; \xi),$$

$$(7) \quad (g(\xi), h) = [C_1(h; \xi)]^- + i[C_2(h; \xi)]^-.$$

Since $Q(\lambda) = i \operatorname{Im} \lambda_0 + (\lambda - \lambda_0) + |\lambda - \lambda_0|^2 \int_{-\infty}^{\infty} (t - \lambda)^{-1} d\rho_0(t)$, and since $\theta(\lambda) = a + b\lambda + \int_{-\infty}^{\infty} (t - \lambda)^{-1} d\rho(t)$, it follows from the lemma that $\theta(\lambda) + Q(\lambda)$ can be defined continuously down to the real axis and that at a point ξ on the real axis, $\theta(\xi) + Q(\xi) = A(\xi) + iB(\xi)$. Since $\rho'_0(\xi) > 0$ for all ξ , $[\theta(\lambda) + Q(\lambda)]^{-1}$ will be continuous down to the real axis, and on the real axis,

$$(8) \quad [\theta(\xi) + Q(\xi)]^{-1} = [A(\xi) + iB(\xi)]^{-1}.$$

From equation (1) and equations (5), (6), (7), (8), we now see that $(R(\lambda)f, h)$ can be defined continuously down to the real axis, where $R(\lambda)$ is the generalized resolvent corresponding to A^+ , and we see that on the real axis,

$$(9) \quad (R(\xi)f, h) = P \int_{-\infty}^{\infty} (t - \xi)^{-1} F(t)[H(t)]^{-\rho'_0(t)} dt \\ + i\pi F(\xi)[H(\xi)]^{-\rho'_0(\xi)} \\ - [A(\xi) + iB(\xi)]^{-1} [C_1(f; \xi) + iC_2(f; \xi)] \\ \cdot \{ [C_1(h; \xi)]^- + i[C_2(h; \xi)]^- \}.$$

The Stieltjes inversion formula states that

$$(10) \quad \begin{aligned} & \left(\left[(1/2) \{ E^+(\beta) + E^+(\beta + 0) \} - (1/2) \{ E^+(\alpha) + E^+(\alpha + 0) \} \right] f, h \right) \\ &= (2\pi i)^{-1} \lim_{\eta \rightarrow +0} \int_{\alpha}^{\beta} [(R(\xi + i\eta)f, h) - (R(\xi + i\eta)h, f)^-] d\xi. \end{aligned}$$

Since the terms under the integral sign are continuous down to the real axis, we may take the limit under the integral sign. Using equation (9), we obtain that

$$(11) \quad \begin{aligned} & (2\pi i)^{-1} \lim_{\eta \rightarrow +0} [(R(\xi + i\eta)f, h) - (R(\xi + i\eta)h, f)^-] \\ &= F(\xi) [H(\xi)]^{-\rho'_0(\xi)} + \sum_{i,j=1}^2 C_i(f; \xi) [C_j(h; \xi)]^{-d_{ij}(\xi)}, \end{aligned}$$

where

$$\begin{aligned} d_{ii}(\xi) &= -d_{22}(\xi) = B(\xi)/\pi [A^2(\xi) + B^2(\xi)], \\ d_{12}(\xi) &= d_{21}(\xi) = -A(\xi)/\pi [A^2(\xi) + B^2(\xi)]. \end{aligned}$$

Now,

$$(f, E_0(\xi)g_0) = \int_{-\infty}^{\xi} F(t) d\rho_0(t).$$

Hence,

$$F(\xi) = (f, E_0(\xi)g_0)' / \rho'_0(\xi) = C_2(f; \xi) / \pi(\xi - \lambda_0^-) \rho'_0(\xi).$$

Similarly,

$$H(\xi) = C_2(h; \xi) / \pi(\xi - \lambda_0^-) \rho'_0(\xi).$$

Substituting these expressions for $F(\xi)$ and $H(\xi)$ into (11) and collecting terms, we obtain that

$$(12) \quad \begin{aligned} & (2\pi i)^{-1} \lim_{\eta \rightarrow +0} [(R(\xi + i\eta)f, h) - (R(\xi + i\eta)h, f)^-] \\ &= \sum_{i,j=1}^2 C_i(f; \xi) [C_j(h; \xi)]^{-\delta_{ij}(\xi)}. \end{aligned}$$

The theorem then follows from equations (10) and (12).

THEOREM 2. *Suppose the conditions of Theorem 1 are valid, with the additional assumption that if A^+ is a selfadjoint dilation of A , then it is minimal (i.e., it is reduced by no nontrivial subspace of $H^+ \ominus H$). If $\rho'(\xi) > 0$ for all ξ , then A^+ is unitarily equivalent to the multiplication operator in $L^2(-\infty, \infty) \oplus L^2(-\infty, \infty)$; hence, the spectral multiplicity of A^+ is two, and the whole real axis is in the continuous spectrum. On*

the other hand, if $\rho'(\xi) \equiv 0$, then A^+ is unitarily equivalent to the multiplication operator in $L^2(-\infty, \infty)$; hence, the spectral multiplicity of A^+ is one, and again the whole real axis is in the continuous spectrum. (In particular, the latter statement holds if A^+ is a selfadjoint extension of A in H .)

PROOF. Let Z be the set of all elements in H^+ of the form $[E^+(\beta) - E^+(\alpha)]f$, where α, β are any two numbers such that $\alpha < \beta$, and f is any element in H . Let Z_1 be the set of elements of the same form except that f is any element of S . Since A^+ is minimal, it is known that H^+ is the closed linear hull of Z . (See Naimark [3].) Since S is dense in H , H^+ is also the closed linear hull of Z_1 .

By completing the square on the integrand, we can write equation (2) in the form

$$\begin{aligned}
 & ([E^+(\beta) - E^+(\alpha)]f, h) \\
 &= \int_{\alpha}^{\beta} [C_1(f; \xi)\delta_{11}^{1/2}(\xi) + \delta_{12}(\xi)\delta_{11}^{-1/2}(\xi)C_2(f; \xi)] \\
 (13) \quad & \cdot [C_1(h; \xi)\delta_{11}^{1/2}(\xi) + \delta_{12}(\xi)\delta_{11}^{-1/2}(\xi)C_2(h; \xi)]^{-} d\xi \\
 &+ \int_{\alpha}^{\beta} C_2(f; \xi)[C_2(h; \xi)]^{-} [\delta_{11}(\xi)\delta_{22}(\xi) - \delta_{12}^2(\xi)]\delta_{11}^{-1}(\xi) d\xi.
 \end{aligned}$$

Let us note that

$$(14) \quad \delta_{11}(\xi)\delta_{22}(\xi) - \delta_{12}^2(\xi) = \rho'(\xi) \{ \pi^2 [A^2(\xi) + B^2(\xi)] \cdot |\xi - \lambda_0|^2 \rho_0'(\xi) \}^{-1}.$$

If $\rho'(\xi) > 0$ for all ξ , then $\delta_{11}(\xi)\delta_{22}(\xi) - \delta_{12}^2(\xi) > 0$ for all ξ . In this case we define an operator V on Z_1 into $L^2(-\infty, \infty) \oplus L^2(-\infty, \infty)$ as follows:

$$\begin{aligned}
 V[E^+(\beta) - E^+(\alpha)]f &= \chi_{[\alpha, \beta]}(\xi) [C_1(f; \xi)\delta_{11}^{1/2}(\xi) + \delta_{12}(\xi)\delta_{11}^{-1/2}(\xi)C_2(f; \xi), \\
 & C_2(f; \xi) \{ \delta_{11}(\xi)\delta_{22}(\xi) - \delta_{12}^2(\xi) \}^{1/2} \delta_{11}^{-1/2}(\xi)],
 \end{aligned}$$

where $\chi_{[\alpha, \beta]}(\xi)$ is the characteristic function of $[\alpha, \beta]$. From (13) we see that

$$\|V[E^+(\beta) - E^+(\alpha)]f\| = \|[E^+(\beta) - E^+(\alpha)]f\|.$$

We now extend V linearly to the linear hull of Z_1 , and by continuity to all of H^+ . V is then an isometry of H^+ into $L^2(-\infty, \infty) \oplus L^2(-\infty, \infty)$.

V is in fact onto, because the linear hull of the set VZ_1 is dense in

$L^2(-\infty, \infty) \oplus L^2(-\infty, \infty)$. To see this, suppose $[k_1(\xi), k_2(\xi)] \in L^2(-\infty, \infty) \oplus L^2(-\infty, \infty)$ and that $[k_1(\xi), k_2(\xi)]$ is perpendicular to VZ_1 . Then,

$$(15) \quad \int_{[\alpha, \beta]} \{ [C_1(f; \xi) \delta_{11}^{1/2}(\xi) + \delta_{12}(\xi) \delta_{11}^{-1/2}(\xi) C_2(f; \xi)] k_1^-(\xi) + C_2(f; \xi) [\delta_{11}(\xi) \delta_{22}(\xi) - \delta_{12}^2(\xi)]^{1/2} \delta_{11}^{-1/2}(\xi) k_2^-(\xi) \} d\xi = 0$$

for all $\alpha, \beta, \alpha < \beta$, and for all $f \in S$. Let (γ, δ) be an arbitrary interval, and let $\{\gamma_n\}, \{\delta_n\}$ be sequences of numbers such that $\gamma > \gamma_1 > \gamma_2 > \dots \rightarrow -\infty$, and $\delta < \delta_1 < \delta_2 < \dots \rightarrow +\infty$. Let $F_n(t)$ be any function in $L^2_{\rho_0}$ which is continuously differentiable, vanishes in $[\gamma_n, \delta_n]$ and is positive outside $[\gamma_n, \delta_n]$. Let f_n be the element in S of which $F_n(t)$ is the transform, and let $g_n = f_n / (f_n, g_0)$. Then, if $G_n(t)$ is the transform of g_n , we have that $\int_{-\infty}^{\infty} G_n(t) d\rho_0(t) = 1$. Since $(g_n, E_0(\xi)g_0) = \int_{-\infty}^{\xi} G_n(t) d\rho_0(t)$, $C_2(g_n; \xi) = \pi(\xi - \lambda_0^-)(g_n, E_0(\xi)g_0)' = 0$ for $\gamma < \xi < \delta$. By equation (6), $C_1(g_n; \xi) = (g_n, g(\xi))$ for $\gamma < \xi < \delta$. Now, $\lim_{n \rightarrow \infty} (g_n, g(\xi)) = 1$ uniformly for $\gamma < \xi < \delta$, because for $\gamma < \xi < \delta$,

$$\begin{aligned} |(g_n, g(\xi)) - 1| &= \left| \int_{-\infty}^{\gamma_n} G_n(t)(t - \lambda_0^-)(t - \xi)^{-1} d\rho_0(t) \right. \\ &\quad \left. + \int_{\delta_n}^{\infty} G_n(t)(t - \lambda_0^-)(t - \xi)^{-1} d\rho_0(t) \right. \\ &\quad \left. - \int_{-\infty}^{\gamma_n} G_n(t) d\rho_0(t) - \int_{\delta_n}^{\infty} G_n(t) d\rho_0(t) \right| \\ &\leq \int_{-\infty}^{\gamma_n} G_n(t) |\xi - \lambda_0^-| |t - \xi|^{-1} d\rho_0(t) \\ &\quad + \int_{\delta_n}^{\infty} G_n(t) |\xi - \lambda_0^-| |t - \xi|^{-1} d\rho_0(t) \\ &\leq K / \min\{\delta_n - \delta, \gamma - \gamma_n\}, \end{aligned}$$

where K is a constant. Thus, we see that for the sequence $\{g_n\}$, $C_2(g_n; \xi) = 0$ for $\gamma < \xi < \delta$, and $\lim_{n \rightarrow \infty} C_1(g_n; \xi) = 1$ uniformly for $\gamma < \xi < \delta$. If we substitute the elements g_n into equation (15) and take the limit as $n \rightarrow \infty$, we shall obtain that $\int_{[\alpha, \beta]} \delta_{11}^{1/2}(\xi) k_1^-(\xi) d\xi = 0$. Since this equation is true for all α, β , it follows that $\delta_{11}^{1/2}(\xi) k_1^-(\xi) = 0$ a.e., and therefore $k_1(\xi) = 0$ a.e. Equation (15) thus becomes

$$(16) \quad \int_{[\alpha, \beta]} C_2(f; \xi) \{ \delta_{11}(\xi) \delta_{22}(\xi) - \delta_{12}^2(\xi) \}^{1/2} \delta_{11}^{-1/2}(\xi) k_2^-(\xi) d\xi = 0$$

for all $\alpha, \beta, \alpha < \beta$, and for all $f \in S$. Now,

$$C_2(g_0; \xi) = \pi(\xi - \lambda_0^-)(g_0, E_0(\xi)g_0)' = \pi(\xi - \lambda_0^-)\rho_0'(\xi).$$

Taking $f = g_0$ in equation (16), we see, then, that

$$\pi(\xi - \lambda_0^-)\rho_0'(\xi) \{ \delta_{11}(\xi)\delta_{22}(\xi) - \delta_{12}^2(\xi) \}^{1/2} \delta_{11}^{-1/2}(\xi) k_2^-(\xi) = 0 \text{ a.e.,}$$

and hence $k_2(\xi) = 0$ a.e. Thus, we have shown that if $[k_1(\xi), k_2(\xi)] \in L^2(-\infty, \infty) \oplus L^2(-\infty, \infty)$ and if $[k_1(\xi), k_2(\xi)]$ is perpendicular to VZ_1 , then $k_1(\xi) = 0$ a.e. and $k_2(\xi) = 0$ a.e. This means that the linear hull of VZ_1 is dense in $L^2(-\infty, \infty) \oplus L^2(-\infty, \infty)$ and establishes our claim that V is onto.

It is not difficult to check now that V carries the spectral function of A^+ into the spectral function of the multiplication operator in $L^2(-\infty, \infty) \oplus L^2(-\infty, \infty)$. Hence, if $\rho'(\xi) > 0$ for all ξ , A^+ is unitarily equivalent to the multiplication operator in $L^2(-\infty, \infty) \oplus L^2(-\infty, \infty)$.

Now suppose that $\rho'(\xi) = 0$ for all ξ . Then by equation (14), $\delta_{11}(\xi)\delta_{22}(\xi) - \delta_{12}^2(\xi) = 0$ for all ξ . Equation (13) becomes

$$\begin{aligned} & ([E^+(\beta) - E^+(\alpha)]f, h) \\ (17) \quad &= \int_{\alpha}^{\beta} [C_1(f; \xi)\delta_{11}^{1/2}(\xi) + \delta_{12}(\xi)\delta_{11}^{-1/2}(\xi)C_2(f; \xi)] \\ & \cdot [C_1(h; \xi)\delta_{11}^{1/2}(\xi) + \delta_{12}(\xi)\delta_{11}^{-1/2}(\xi)C_2(h; \xi)]^- d\xi. \end{aligned}$$

In this case we define an operator V on Z_1 into $L^2(-\infty, \infty)$ as follows:

$$V[E^+(\beta) - E^+(\alpha)]f = \chi_{[\alpha, \beta)}(\xi) [C_1(f; \xi)\delta_{11}^{1/2}(\xi) + \delta_{12}(\xi)\delta_{11}^{-1/2}(\xi)C_2(f; \xi)].$$

Proceeding as above, we can use V to show that A^+ is unitarily equivalent to the multiplication operator in $L^2(-\infty, \infty)$. If A^+ is a selfadjoint extension of A in H , then $\rho(\xi) \equiv 0$, and the above holds.

This completes the proof of Theorem 2.

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CALIFORNIA STATE COLLEGE AT FULLERTON