SYMMETRIC OPERATORS WITH TWICE CONTINUOUSLY DIFFERENTIABLE SPECTRAL FUNCTIONS

RICHARD C. GILBERT[1]

In a previous article [2] it was shown that if A is a simple closed symmetric operator with deficiency indices (1, 1) in a Hilbert space H and if A has a selfadjoint extension A_0 in H with a discrete spectrum, then every selfadjoint extension and every minimal selfadjoint dilation A^+ of A has spectral multiplicity 1. Further, it was possible to make certain statements about the spectrum of A^+. In the present article it is shown that if A has a selfadjoint extension A_0 with a twice continuously differentiable spectral function, whose derivative is always positive, then there are minimal selfadjoint dilations A^+ of A for which the spectral multiplicity is two; indeed, there are minimal selfadjoint dilations A^+ which are unitarily equivalent to the multiplication operator in $L^2(-\infty, \infty)\oplus L^2(-\infty, \infty)$. As in [2] the proof of these facts depends upon obtaining an expansion theorem which is valid for each of the selfadjoint extensions or dilations A^+. This expansion theorem is the analog of the well-known expansion theorems for linear ordinary differential operators; however, it is necessary in deriving it to overcome the fact that, unlike a linear ordinary differential operator, there does not necessarily exist a basis of entire functions for the solutions of the equation $Af=\lambda f$. After the expansion theorem is obtained, it is used to define a unitary map of H^+ (the space in which A^+ acts) onto $L^2(-\infty, \infty)\oplus L^2(-\infty, \infty)$. The same map takes A^+ into the multiplication operator in $L^2(-\infty, \infty)$ $\oplus L^2(-\infty, \infty)$.

Throughout this article we shall adhere to the notation of [2], except that we shall use the symbol $Q(\lambda)$ in place of $Q_1(\lambda)$.

We begin with a lemma on the limit of a function analytic in the upper half-plane as one approaches the real axis.

LEMMA. *Suppose for all nonreal* λ, $\theta(\lambda)=\int_{-\infty}^{\infty}(t-\lambda)^{-1}d\rho(t)$, *where* $\rho(t)$ *is a nondecreasing bounded function which is twice continuously differentiable for all* t. *Then,* $\theta(\lambda)$ *may be defined continuously down to the*

Presented to the Society, April 26, 1969; received by the editors February 10, 1969.

[1] This work was performed in part while the author was on a research leave granted by California State College at Fullerton.

real axis. For a point ξ on the real axis we shall have

$$\operatorname{Re} \theta(\xi) = P \int_{-\infty}^{\infty} (t - \xi)^{-1} d\rho(t),$$

$$\operatorname{Im} \theta(\xi) = \pi \rho'(\xi),$$

where P stands for Cauchy principal value.

Proof. Let $f(\xi, t) = [\rho'(t) - \rho'(\xi)](t-\xi)^{-1}$ if $t \neq \xi$, and let $f(\xi, t) = p''(t)$ if $t = \xi$. Then, $f(\xi, t)$ is a continuous function, and $\rho'(t) = \rho'(\xi) + (t-\xi)f(\xi, t)$.

Now,

$$\begin{aligned} P \int_{-\infty}^{\infty} & (t - \xi)^{-1} d\rho(t) \\ &= \int_{-\infty}^{\xi-1} (t - \xi)^{-1} d\rho(t) + \int_{\xi+1}^{\infty} (t - \xi)^{-1} d\rho(t) + \int_{\xi-1}^{\xi+1} f(\xi, t) dt \\ &= \int_{-\infty}^{-1} u^{-1} \rho'(u + \xi) du + \int_{1}^{\infty} u^{-1} \rho'(u + \xi) du + \int_{-1}^{1} f(\xi, u + \xi) du. \end{aligned}$$

From this last expression it is not difficult to see that $P\int_{-\infty}^{\infty}(t-\xi)^{-1}d\rho(t)$ exists and is a continuous function of ξ. If $\lambda = \xi + i\eta$, then

$$\begin{aligned} \operatorname{Re} \theta(\lambda) &= \int_{-\infty}^{\infty} (t - \xi)[(t - \xi)^2 + \eta^2]^{-1} d\rho(t) \\ &= \int_{-\infty}^{-1} [u + (\xi_0 - \xi)]\{[u + (\xi_0 - \xi)]^2 + \eta^2\}^{-1} \rho'(u + \xi_0) du \\ &\quad + \int_{1}^{\infty} [u + (\xi_0 - \xi)]\{[u + (\xi_0 - \xi)]^2 + \eta^2\}^{-1} \rho'(u + \xi_0) du \\ &\quad + (1/2)\rho'(\xi) \log\{[1 - (\xi - \xi_0)]^2 + \eta^2\}\{[1 + (\xi - \xi_0)]^2 + \eta^2\}^{-1} \\ &\quad + \int_{-1}^{1} f(\xi, u + \xi_0) du - \eta^2 \int_{-1}^{1} \{[u - (\xi - \xi_0)]^2 + \eta^2\}^{-1} \\ &\quad \cdot f(\xi, u + \xi_0) du. \end{aligned}$$

From this last expression it is not difficult to see that $\lim_{\lambda \to \xi_0} \operatorname{Re} \theta(\lambda) = P\int_{-\infty}^{\infty}(t-\xi_0)^{-1}d\rho(t)$. Hence, $\operatorname{Re} \theta(\lambda)$ can be defined continuously down to the real axis, and its value on the real axis is $P\int_{-\infty}^{\infty}(t-\xi)^{-1}d\rho(t)$.

Now,

$$
\begin{aligned}
\operatorname{Im} \theta(\lambda) &= \eta \int_{-\infty}^{\infty} [(t-\xi)^2 + \eta^2]^{-1} d\rho(t) \\
&= \eta \int_{-\infty}^{-1} \{[u - (\xi - \xi_0)]^2 + \eta^2\}^{-1} \rho'(u + \xi_0) du \\
&\quad + \eta \int_{1}^{\infty} \{[u - (\xi - \xi_0)]^2 + \eta^2\}^{-1} \rho'(u + \xi_0) du \\
&\quad + \rho'(\xi)\{\tan^{-1}\eta^{-1}[1 - (\xi - \xi_0)] + \tan^{-1}\eta^{-1}[1 + (\xi - \xi_0)]\} \\
&\quad + \eta \int_{-1}^{1} [u - (\xi - \xi_0)]\{[u - (\xi - \xi_0)]^2 + \eta^2\}^{-1} f(\xi, u + \xi_0) du.
\end{aligned}
$$

From this expression it is not difficult to see that $\lim_{\lambda \to \xi_0} \operatorname{Im} \theta(\lambda) = \pi\rho'(\xi_0)$. Since $\rho'(\xi)$ is continuous, we see that $\operatorname{Im} \theta(\lambda)$ can be defined continuously down to the real axis, and its value on the real axis is $\pi\rho'(\xi)$.

This completes the proof of the lemma.

Let A be a simple closed symmetric operator with deficiency indices (1, 1) in the Hilbert space H. Let λ_0 be a complex number with positive imaginary part, and let g_0 be an element of norm 1 in the deficiency subspace of A, corresponding to λ_0^-. (Here λ_0^- stands for complex conjugate.) Suppose that A_0 is a selfadjoint extension of A in H. Since A is simple, g_0 is a generating element for A_0. (See Achieser and Glasmann [1, §81, Theorem 2].) Hence, A_0 is unitarily equivalent to the multiplication operator in $L^2_{\rho_0}$, where $\rho_0(t) = (E_0(t)g_0, g_0)$ and $E_0(t)$ is the spectral function of A_0. By S we shall denote those elements f in H whose transforms $F(t)$ in $L^2_{\rho_0}$ are continuously differentiable functions. S is dense in H.

Suppose that $R_0(\lambda)$ is the resolvent of A_0. Let $g(\lambda) = g_0 + (\lambda - \lambda_0) \cdot R_0(\lambda) g_0$, and let $Q(\lambda) = i \operatorname{Im} \lambda_0 + (\lambda - \lambda_0)(g_0, g(\lambda^-))$. As is indicated in [2, Lemma 2], for $\operatorname{Im} \lambda \neq 0$ the generalized resolvent $R(\lambda)$ of A corresponding to a selfadjoint extension or dilation A^+ of A has the form

$$
R(\lambda) = R_0(\lambda) - [\theta(\lambda) + Q(\lambda)]^{-1}(\cdot, g(\lambda^-)) g(\lambda), \tag{1}
$$

where $\theta(\lambda)$ is an analytic function for $\operatorname{Im} \lambda \neq 0$ which has nonnegative imaginary part in the upper half-plane. $R(\lambda)$ corresponds to a selfadjoint extension in the space H if and only if $\theta(\lambda)$ is identically equal to a constant θ, $-\infty < \theta \leqq +\infty$.

THEOREM 1 (EXPANSION THEOREM). *Suppose $\rho_0(t)$ is twice continuously differentiable and that $\rho_0'(t) > 0$ for all t. Let A^+ be a selfadjoint* *extension of A in H or a selfadjoint dilation of A in $H^+ \supset H$. Let $E^+(\lambda)$ be the spectral function of A^+. Suppose the function $\theta(\lambda)$ of equation* (1) *has the form $\theta(\lambda)=a+b\lambda+\int_{-\infty}^{\infty}(t-\lambda)^{-1}d\rho(t)$, where $b\geqq 0$, a is real, $\rho(t)$ is nondecreasing, bounded and twice continuously differentiable. Then for any numbers α, β, $\alpha<\beta$, and for any two elements f, $h\in S$,*

$$\text{(2)}\qquad ([E^+(\beta) - E^+(\alpha)]f, h) = \int_\alpha^\beta \sum_{i,j=1}^{2} C_i(f;\xi)[C_j(h;\xi)]^-\delta_{ij}(\xi)d\xi,$$

where

$$C_1(f;\xi) = (f, g_0) + (\xi - \lambda_0^-)P\int_{-\infty}^{\infty}(t-\xi)^{-1}d(f, E_0(t)g_0),$$

$$C_2(f;\xi) = \pi(\xi - \lambda_0^-)(f, E_0(\xi)g_0)', \quad \textit{where the prime stands for differentiation},$$

$$\delta_{11}(\xi) = B(\xi)/\pi[A^2(\xi) + B^2(\xi)],$$

$$\delta_{12}(\xi) = \delta_{21}(\xi) = -A(\xi)/\pi[A^2(\xi) + B^2(\xi)],$$

$$\delta_{22}(\xi) = 1/\pi^2|\xi - \lambda_0|^2\rho_0'(\xi) - B(\xi)/\pi[A^2(\xi) + B^2(\xi)],$$

$$A(\xi) = P\int_{-\infty}^{\infty}(t-\xi)^{-1}d\rho(t) + a + b\xi + (\xi - \operatorname{Re}\lambda_0) + |\xi - \lambda_0|^2P\int_{-\infty}^{\infty}(t-\xi)^{-1}d\rho_0(t),$$

$$B(\xi) = \pi[\rho'(\xi) + |\xi - \lambda_0|^2\rho_0'(\xi)].$$

PROOF. By the process of polarization applied to $(E_0(t)f, h)$, we see that

$$\begin{aligned}(R_0(\lambda)f, h) &= \int_{-\infty}^{\infty}(t-\lambda)^{-1}d(E_0(t)f, h)\\ &= (1/4)\int_{-\infty}^{\infty}(t-\lambda)^{-1}d(E_0(t)(f+h), f+h)\\ &\quad - (1/4)\int_{-\infty}^{\infty}(t-\lambda)^{-1}d(E_0(t)(f-h), f-h)\\ &\quad + (i/4)\int_{-\infty}^{\infty}(t-\lambda)^{-1}d(E_0(t)(f+ih), f+ih)\\ &\quad - (i/4)\int_{-\infty}^{\infty}(t-\lambda)^{-1}d(E_0(t)(f-ih), f-ih).\end{aligned}\qquad (3)$$

Since $(E_0(t)(f+h), f+h)=\int_{-\infty}^{t}|F(\xi)+H(\xi)|^2 d\rho_0(\xi)$, and since $F(\xi)$, $H(\xi)$ are continuously differentiable, we see that $(E_0(t)(f+h), f+h) \in C^2$. The same is true for the other terms in (3). It follows from the lemma, then, that $(R_0(\lambda)f, h)$ can be defined continuously down to the real axis and that at a point ξ on the real axis,

$$(4) \qquad (R_0(\xi)f, h) = P\int_{-\infty}^{\infty}(t-\xi)^{-1}d(E_0(t)f, h) + i\pi(E_0(\xi)f, h)'.$$

Since $(E_0(t)f, h)=\int_{-\infty}^{t}F(u)[H(u)]^{-}\rho_0'(u)du$, equation (4) can be written

$$(5) \qquad \begin{aligned}(R_0(\xi)f, h) = P\int_{-\infty}^{\infty}&(t-\xi)^{-1}F(t)[H(t)]^{-}\rho_0'(t)dt \\ &+ i\pi F(\xi)[H(\xi)]^{-}\rho_0'(\xi).\end{aligned}$$

One can show similarly that $(f, g(\lambda^-))$ and $(g(\lambda), h)$ can be defined continuously down to the real axis and that at a point ξ on the real axis,

$$(6) \qquad (f, g(\xi)) = C_1(f;\xi) + iC_2(f;\xi),$$

$$(7) \qquad (g(\xi), h) = [C_1(h;\xi)]^{-} + i[C_2(h;\xi)]^{-}.$$

Since $Q(\lambda)=i\ \mathrm{Im}\ \lambda_0+(\lambda-\lambda_0)+|\lambda-\lambda_0|^2\int_{-\infty}^{\infty}(t-\lambda)^{-1}d\rho_0(t)$, and since $\theta(\lambda)=a+b\lambda+\int_{-\infty}^{\infty}(t-\lambda)^{-1}d\rho(t)$, it follows from the lemma that $\theta(\lambda)+Q(\lambda)$ can be defined continuously down to the real axis and that at a point ξ on the real axis, $\theta(\xi)+Q(\xi)=A(\xi)+iB(\xi)$. Since $\rho_0'(\xi)>0$ for all ξ, $[\theta(\lambda)+Q(\lambda)]^{-1}$ will be continuous down to the real axis, and on the real axis,

$$(8) \qquad [\theta(\xi)+Q(\xi)]^{-1} = [A(\xi)+iB(\xi)]^{-1}.$$

From equation (1) and equations (5), (6), (7), (8), we now see that $(R(\lambda)f, h)$ can be defined continuously down to the real axis, where $R(\lambda)$ is the generalized resolvent corresponding to A^+, and we see that on the real axis,

$$(9) \qquad \begin{aligned}(R(\xi)f, h) = P\int_{-\infty}^{\infty}&(t-\xi)^{-1}F(t)[H(t)]^{-}\rho_0'(t)dt \\ &+ i\pi F(\xi)[H(\xi)]^{-}\rho_0'(\xi) \\ &- [A(\xi)+iB(\xi)]^{-1}[C_1(f;\xi)+iC_2(f;\xi)] \\ &\cdot\{[C_1(h;\xi)]^{-} + i[C_2(h;\xi)]^{-}\}.\end{aligned}$$

The Stieltjes inversion formula states that

$$\begin{aligned}(10)\quad &([(1/2)\{E^+(\beta) + E^+(\beta+0)\} - (1/2)\{E^+(\alpha) + E^+(\alpha+0)\}]f, h)\\ &= (2\pi i)^{-1} \lim_{\eta\to+0} \int_\alpha^\beta [(R(\xi + i\eta)f, h) - (R(\xi+i\eta)h, f)^-]d\xi.\end{aligned}$$

Since the terms under the integral sign are continuous down to the real axis, we may take the limit under the integral sign. Using equation (9), we obtain that

$$\begin{aligned}(11)\quad &(2\pi i)^{-1} \lim_{\eta\to+0} [(R(\xi + i\eta)f, h) - (R(\xi+i\eta)h, f)^-]\\ &= F(\xi)[H(\xi)]^-\rho_0'(\xi) + \sum_{i,j=1}^{2} C_i(f;\xi)[C_j(h;\xi)]^- d_{ij}(\xi),\end{aligned}$$

where

$$\begin{aligned} d_{ii}(\xi) &= -d_{22}(\xi) = B(\xi)/\pi[A^2(\xi) + B^2(\xi)],\\ d_{12}(\xi) &= d_{21}(\xi) = -A(\xi)/\pi[A^2(\xi) + B^2(\xi)].\end{aligned}$$

Now,

$$(f, E_0(\xi)g_0) = \int_{-\infty}^{\xi} F(t)d\rho_0(t).$$

Hence,

$$F(\xi) = (f, E_0(\xi)g_0)'/\rho_0'(\xi) = C_2(f;\xi)/\pi(\xi - \lambda_0^-)\rho_0'(\xi).$$

Similarly,

$$H(\xi) = C_2(h;\xi)/\pi(\xi - \lambda_0^-)\rho_0'(\xi).$$

Substituting these expressions for $F(\xi)$ and $H(\xi)$ into (11) and collecting terms, we obtain that

$$\begin{aligned}(12)\quad &(2\pi i)^{-1} \lim_{\eta\to+0} [(R(\xi + i\eta)f, h) - (R(\xi+i\eta)h, f)^-]\\ &= \sum_{i,j=1}^{2} C_i(f;\xi)[C_j(h;\xi)]^- \delta_{ij}(\xi).\end{aligned}$$

The theorem then follows from equations (10) and (12).

THEOREM 2. *Suppose the conditions of Theorem* 1 *are valid, with the additional assumption that if A^+ is a selfadjoint dilation of A, then it is minimal (i.e., it is reduced by no nontrivial subspace of $H^+\ominus H$). If $\rho'(\xi) > 0$ for all ξ, then A^+ is unitarily equivalent to the multiplication operator in $L^2(-\infty,\infty)\oplus L^2(-\infty,\infty)$; hence, the spectral multiplicity of A^+ is two, and the whole real axis is in the continuous spectrum. On* *the other hand, if* $\rho'(\xi)\equiv 0$, *then* A^+ *is unitarily equivalent to the multiplication operator in* $L^2(-\infty,\infty)$; *hence, the spectral multiplicity of* A^+ *is one, and again the whole real axis is in the continuous spectrum.* (*In particular, the latter statement holds if* A^+ *is a selfadjoint extension of* A *in* H.)

Proof. Let Z be the set of all elements in H^+ of the form $[E^+(\beta)-E^+(\alpha)]f$, where α, β are any two numbers such that $\alpha<\beta$, and f is any element in H. Let Z_1 be the set of elements of the same form except that f is any element of S. Since A^+ is minimal, it is known that H^+ is the closed linear hull of Z. (See Naimark [**3**].) Since S is dense in H, H^+ is also the closed linear hull of Z_1.

By completing the square on the integrand, we can write equation (2) in the form

$$\begin{aligned}([E^+(\beta) - E^+(\alpha)]f, h) &= \int_\alpha^\beta [C_1(f;\xi)\delta_{11}^{1/2}(\xi) + \delta_{12}(\xi)\delta_{11}^{-1/2}(\xi)C_2(f;\xi)] \\ &\quad\cdot[C_1(h;\xi)\delta_{11}^{1/2}(\xi) + \delta_{12}(\xi)\delta_{11}^{-1/2}(\xi)C_2(h;\xi)]^- d\xi \\ &\quad+ \int_\alpha^\beta C_2(f;\xi)[C_2(h;\xi)]^-[\delta_{11}(\xi)\delta_{22}(\xi) - \delta_{12}^2(\xi)]\delta_{11}^{-1}(\xi)d\xi.\end{aligned} \tag{13}$$

Let us note that

$$\delta_{11}(\xi)\delta_{22}(\xi) - \delta_{12}^2(\xi) = \rho'(\xi)\{\pi^2[A^2(\xi) + B^2(\xi)]\cdot|\xi-\lambda_0|^2\rho_0'(\xi)\}^{-1}. \tag{14}$$

If $\rho'(\xi)>0$ for all ξ, then $\delta_{11}(\xi)\delta_{22}(\xi)-\delta_{12}^2(\xi)>0$ for all ξ. In this case we define an operator V on Z_1 into $L^2(-\infty,\infty)\oplus L^2(-\infty,\infty)$ as follows:

$$\begin{aligned}V[E^+(\beta) - E^+(\alpha)]f = \chi_{[\alpha,\beta)}(\xi)[C_1(f;\xi)\delta_{11}^{1/2}(\xi) + \delta_{12}(\xi)\delta_{11}^{-1/2}(\xi)C_2(f;\xi),\\ C_2(f;\xi)\{\delta_{11}(\xi)\delta_{22}(\xi) - \delta_{12}^2(\xi)\}^{1/2}\delta_{11}^{-1/2}(\xi)],\end{aligned}$$

where $\chi_{[\alpha,\beta)}(\xi)$ is the characteristic function of $[\alpha,\beta)$. From (13) we see that

$$\|V[E^+(\beta) - E^+(\alpha)]f\| = \|[E^+(\beta) - E^+(\alpha)]f\|.$$

We now extend V linearly to the linear hull of Z_1, and by continuity to all of H^+. V is then an isometry of H^+ into $L^2(-\infty,\infty)\oplus L^2(-\infty,\infty)$.

V is in fact onto, because the linear hull of the set VZ_1 is dense in

$L^2(-\infty, \infty)\oplus L^2(-\infty, \infty)$. To see this, suppose $[k_1(\xi), k_2(\xi)] \in L^2(-\infty, \infty)\oplus L^2(-\infty, \infty)$ and that $[k_1(\xi), k_2(\xi)]$ is perpendicular to VZ_1. Then,

$$(15)\quad \int_{[\alpha,\beta)} \{[C_1(f;\xi)\delta_{11}^{1/2}(\xi) + \delta_{12}(\xi)\delta_{11}^{-1/2}(\xi)C_2(f;\xi)]\bar{k_1}(\xi) + C_2(f;\xi)[\delta_{11}(\xi)\delta_{22}(\xi) - \delta_{12}^2(\xi)]^{1/2}\delta_{11}^{-1/2}(\xi)\bar{k_2}(\xi)\}d\xi = 0$$

for all α, β, $\alpha<\beta$, and for all $f\in S$. Let (γ, δ) be an arbitrary interval, and let $\{\gamma_n\}$, $\{\delta_n\}$ be sequences of numbers such that $\gamma>\gamma_1>\gamma_2 > \cdots \to -\infty$, and $\delta<\delta_1<\delta_2< \cdots \to +\infty$. Let $F_n(t)$ sbe any function in $L^2_{\rho_0}$ which is continuously differentiable, vanishes in $[\gamma_n, \delta_n]$ and is positive outside $[\gamma_n, \delta_n]$. Let f_n be the element in S of which $F_n(t)$ is the transform, and let $g_n = f_n/(f_n, g_0)$. Then, if $G_n(t)$ is the transform of g_n, we have that $\int_{-\infty}^{\infty} G_n(t)d\rho_0(t) = 1$. Since $(g_n, E_0(\xi)g_0) = \int_{-\infty}^{\xi} G_n(t)d\rho_0(t)$, $C_2(g_n;\xi) = \pi(\xi-\bar{\lambda_0})(g_n, E_0(\xi)g_0)' = 0$ for $\gamma<\xi<\delta$. By equation (6), $C_1(g_n;\xi) = (g_n, g(\xi))$ for $\gamma<\xi<\delta$. Now, $\lim_{n\to\infty}(g_n, g(\xi)) = 1$ uniformly for $\gamma<\xi<\delta$, because for $\gamma<\xi<\delta$,

$$\begin{aligned} |(g_n, g(\xi)) - 1| &= \Big| \int_{-\infty}^{\gamma_n} G_n(t)(t-\bar{\lambda_0})(t-\xi)^{-1}d\rho_0(t) \\ &\quad + \int_{\delta_n}^{\infty} G_n(t)(t-\bar{\lambda_0})(t-\xi)^{-1}d\rho_0(t) \\ &\quad - \int_{-\infty}^{\gamma_n} G_n(t)d\rho_0(t) - \int_{\delta_n}^{\infty} G_n(t)d\rho_0(t)\Big| \\ &\leqq \int_{-\infty}^{\gamma_n} G_n(t)\,|\xi-\bar{\lambda_0}|\,|t-\xi|^{-1}d\rho_0(t) \\ &\quad + \int_{\delta_n}^{\infty} G_n(t)\,|\xi-\bar{\lambda_0}|\,|t-\xi|^{-1}d\rho_0(t) \\ &\leqq K/\min\{\delta_n-\delta, \gamma-\gamma_n\}, \end{aligned}$$

where K is a constant. Thus, we see that for the sequence $\{g_n\}$, $C_2(g_n;\xi)=0$ for $\gamma<\xi<\delta$, and $\lim_{n\to\infty} C_1(g_n;\xi)=1$ uniformly for $\gamma<\xi<\delta$. If we substitute the elements g_n into equation (15) and take the limit as $n\to\infty$, we shall obtain that $\int_{[\alpha,\beta)} \delta_{11}^{1/2}(\xi)\bar{k_1}(\xi)d\xi = 0$. Since this equation is true for all α, β, it follows that $\delta_{11}^{1/2}(\xi)\bar{k_1}(\xi)=0$ a.e., and therefore $k_1(\xi)=0$ a.e. Equation (15) thus becomes

$$(16)\qquad \int_{[\alpha,\beta)} C_2(f;\xi)\{\delta_{11}(\xi)\delta_{22}(\xi) - \delta_{12}^2(\xi)\}^{1/2}\delta_{11}^{-1/2}(\xi)\bar{k_2}(\xi)d\xi = 0$$

for all α, β, $\alpha<\beta$, and for all $f\in S$. Now,

$$C_2(g_0;\xi) = \pi(\xi-\bar{\lambda}_0)(g_0, E_0(\xi)g_0)' = \pi(\xi-\bar{\lambda}_0)\rho_0'(\xi).$$

Taking $f=g_0$ in equation (16), we see, then, that

$$\pi(\xi-\bar{\lambda}_0)\rho_0'(\xi)\left\{\delta_{11}(\xi)\delta_{22}(\xi)-\delta_{12}^2(\xi)\right\}^{1/2}\delta_{11}^{-1/2}(\xi)\bar{k}_2(\xi) = 0 \text{ a.e.},$$

and hence $k_2(\xi)=0$ a.e. Thus, we have shown that if $[k_1(\xi), k_2(\xi)]$ $\in L^2(-\infty,\infty)\oplus L^2(-\infty,\infty)$ and if $[k_1(\xi), k_2(\xi)]$ is perpendicular to VZ_1, then $k_1(\xi)=0$ a.e. and $k_2(\xi)=0$ a.e. This means that the linear hull of VZ_1 is dense in $L^2(-\infty,\infty)\oplus L^2(-\infty,\infty)$ and establishes our claim that V is onto.

It is not difficult to check now that V carries the spectral function of A^+ into the spectral function of the multiplication operator in $L^2(-\infty,\infty)\oplus L^2(-\infty,\infty)$. Hence, if $\rho'(\xi)>0$ for all ξ, A^+ is unitarily equivalent to the multiplication operator in $L^2(-\infty,\infty)$ $\oplus L^2(-\infty,\infty)$.

Now suppose that $\rho'(\xi)=0$ for all ξ. Then by equation (14), $\delta_{11}(\xi)\delta_{22}(\xi)-\delta_{12}^2(\xi)=0$ for all ξ. Equation (13) becomes

$$\begin{aligned}(17)\qquad &([E^+(\beta)-E^+(\alpha)]f, h)\\ &= \int_\alpha^\beta [C_1(f;\xi)\delta_{11}^{1/2}(\xi)+\delta_{12}(\xi)\delta_{11}^{-1/2}(\xi)C_2(f;\xi)]\\ &\quad\cdot \overline{[C_1(h;\xi)\delta_{11}^{1/2}(\xi)+\delta_{12}(\xi)\delta_{11}^{-1/2}(\xi)C_2(h;\xi)]}\,d\xi.\end{aligned}$$

In this case we define an operator V on Z_1 into $L^2(-\infty,\infty)$ as follows:

$$V[E^+(\beta)-E^+(\alpha)]f = \chi_{[\alpha,\beta)}(\xi)[C_1(f;\xi)\delta_{11}^{1/2}(\xi)+\delta_{12}(\xi)\delta_{11}^{-1/2}(\xi)C_2(f;\xi)].$$

Proceeding as above, we can use V to show that A^+ is unitarily equivalent to the multiplication operator in $L^2(-\infty,\infty)$. If A^+ is a selfadjoint extension of A in H, then $\rho(\xi)\equiv 0$, and the above holds.

This completes the proof of Theorem 2.

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California State College at Fullerton