HOMOGENEOUS ALMOST TANGENT STRUCTURES

M. P. CLOSS

1. **Introduction.** Let J be a field of linear operators acting on the complexified tangent spaces of a differentiable manifold V_{2n}, of dimension $2n$, and which satisfies a relation of the form $J^2=\lambda^2 I$ where λ is a complex constant and I is the identity operator. In the case $\lambda\neq 0$ the manifold has an almost product structure which in the case $\lambda=i$ reduces to an almost complex structure. In the remaining case, $\lambda=0$, (rank $J=n$) the manifold has an almost tangent structure [1]. In [2] Frolicher has studied homogeneous almost complex structures and in [3] Legrand has studied homogeneous almost product structures. In the present paper the author extends this treatment to the almost tangent structures.

2. **Notation.** Let G be a connected Lie group and H a closed subgroup. The quotient space G/H (under the equivalence relation $g_1^{-1}g_2\in H$) is able to be endowed with the structure of a real analytic manifold V_m with an analytic projection $p: G\to G/H$. G has with respect to p, a natural structure of a principal fibre bundle with base space V_m and structural group H effective on itself by left translation.

Let us be given an element $\xi\in G$. To the point $x=pg$ (where $g\in G$) of V_m, let us correspond the point

$$K_\xi x = p(\xi g).$$

$K_\xi x$ is independent of the choice of g in $p^{-1}(x)$. G thus appears as a transitive group of analytic transformations of V_m. We will say that V_m, furnished with the structure defined by the group of operators G, is a homogeneous Lie space.

Let the differential of K_ξ at x be denoted by $K'_\xi(x)$. Then $K'_\xi(x)$ is a linear mapping of the tangent vector space at x onto the tangent vector space at $K_\xi x$. In particular at the point $x_0=pe$ (where e is the identity of G) and for $h\in H$ we have $K'_h(x_0)$ is a linear mapping of T_{x_0} into itself. We may thus represent H by a group $\tilde{H}$ of linear transformations of T_{x_0}. We shall briefly denote $K'_\xi(x_0)$ by K'_ξ.

If $p': T_g\to T_{pg}$ denotes the linear mapping of the tangent vector spaces of G onto the tangent vector spaces of V_m we will have for any $u\in T_e$

$$K'_h\, p'u = p'\,(\operatorname{adj} h)u$$

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where (adj h) is the element corresponding to $h \in H$ under the adjoint representation of G.

The extension of the mapping $K'_\xi(x)$ to the tensors defined on T_x will be represented by the same symbol. In particular, for the case of a tensor J of type (1, 1) defined on T_{x_0} we will have

$$K'_\xi(J) = K'_{\xi^{-1}} \circ J \circ K'_\xi .$$

We will assume in the sequel that V_m is of dimension $m = 2n$ for some n.

3. **Homogeneous almost tangent structures.** Let us consider on a homogeneous Lie space $V_{2n} = G/H$ a field of real or complex tensors; let t_x be the tensor of the field at the point $x = pg$. A necessary and sufficient condition that the field be invariant under G is that

$$t_x = K'_g(t_{x_0}) \tag{1}$$

for an arbitrary $g \in p^{-1}(x_0)$, that is, be sent into itself by each element of the group G. Clearly, t_{x_0} is invariant under $\tilde{H}$. From a tensor defined at the point x_0 and invariant under $\tilde{H}$, we may use the formula (1) to define on V_m a field of tensors invariant under G.

In particular let us suppose that there exists at the point x_0 a tensor (F_i^j), invariant under $\tilde{H}$ and of rank n such that

$$F_i^j F_j^k = 0. \tag{2}$$

One deduces from this by the formula (1) a field of tensors invariant under G and which satisfies the relation (2) at each point of V_{2n}. It thus determines an almost tangent structure on V_{2n}. We will then say that V_{2n} is furnished with a homogeneous almost tangent structure.

If J is a (1, 1) tensor defined on T_{x_0} then it will be invariant under H if and only if

$$K'_h(J) = J$$

that is

$$K'_{h^{-1}} \circ J \circ K'_h = J$$

or

$$J \circ K'_h = K'_h \circ J \quad \text{for any } h \in H.$$

$T^c_{x_0}$ being the complexified vector space of T_{x_0}, a homogeneous almost tangent structure is able to be defined by the data on $T^c_{x_0}$ of a linear operator J, of rank n, satisfying the following conditions:

(3) $$J^2 = 0$$

(4) $$J \circ K'_h = K'_h \circ J \quad \text{for any } h \in H.$$

(3) is the translation of (2) in terms of operators and (4) expresses the invariance of J under $\tilde{H}$.

Let us denote by $\boldsymbol{G}$ the tangent vector space to G at the point e, furnished with its Lie algebra structure; by $\boldsymbol{G}^c$ its complexification; and by $\boldsymbol{H}$ and $\boldsymbol{H}^c$ the analogous spaces relative to the subgroup H of G. Let us choose in $\boldsymbol{G}^c$ a subspace M supplementary to $\boldsymbol{H}^c$, that is such that $\boldsymbol{G}^c = \boldsymbol{H}^c \oplus M$. Since any $\boldsymbol{R}$-linear map between real vector spaces extends in a unique way to a $\boldsymbol{C}$-linear map of the complexified spaces, the mapping $p': \boldsymbol{G} \to T_{x_0}$ is able to be extended to $\boldsymbol{G}^c$. By restriction to M it determines an isomorphism of M onto $T^c_{x_0}$, for which we denote the inverse isomorphism by q'.

To any linear operator J defined on $T^c_{x_0}$ one is able to associate the linear operator $I: \boldsymbol{G}^c \to \boldsymbol{G}^c$ defined by setting $Iu = q'Jp'u$ for any $u \in G^c$.

If J is of rank n on $T^c_{x_0}$, then $J(T^c_{x_0})$ is of dimension n and since q' is an isomorphism of $T^c_{x_0}$ onto M then $I(\boldsymbol{G}^c)$ is an n-dimensional subspace of M. Thus I is a linear operator of rank n, zero on $\boldsymbol{H}^c$, and is such that $I(\boldsymbol{G}^c) \cap \boldsymbol{H}^c = \boldsymbol{O}$. We note that $p'q'$ is the identity on $J(T^{c^*}_x)$ and $q'p'$ is the identity on M.

Let $h \in H$; one then has for any $u \in \boldsymbol{G}^c$

$$\begin{aligned} p'(\text{adj } h)Iu &= K'_h p'Iu \\ &= K'_h p'q'Jp'u \\ &= K'_h Jp'u \end{aligned}$$

and

$$\begin{aligned} p'I(\text{adj } h)u &= p'q'Jp'(\text{adj } h)u \\ &= Jp'(\text{adj } h)u \\ &= JK'_h p'u. \end{aligned}$$

Hence

(5) $$p'((\text{adj } h)I - I(\text{adj } h))u = (K'_h J - JK'_h)p'u.$$

Suppose now that J satisfies (3) and (4); then for any $u \in \boldsymbol{G}^c$ the operator I associated to J is such that:

$$I^2u = q'Jp'q'Jp'u = q'J^2p'u = 0$$

and according to (5) such that:

$$p'((\text{adj } h)I - I(\text{adj } h))\boldsymbol{u} = 0$$

for any $h \in H$. One is then able to write

$$I^2 u = 0 \tag{6}$$

$$(\text{adj } h)Iu \equiv I(\text{adj } h)u \quad (\text{modulo } \boldsymbol{H}^c) \tag{7}$$

for any $u \in \boldsymbol{G}^c$ and $h \in H$.

Conversely, let I be a linear operator of $\boldsymbol{G}^c$ into itself such that $I(\boldsymbol{G}^c) \cap \boldsymbol{H}^c = \boldsymbol{O}$ where I is zero on $\boldsymbol{H}^c$, of rank n, and satisfies (6) and (7). Then $I(\boldsymbol{G}^c)$ is an n-dimensional subspace of M where M is supplementary to $\boldsymbol{H}^c$ in $\boldsymbol{G}^c$. Let q' be, as earlier, the isomorphism of $T^c_{x_0}$ onto M. We define a linear operator J of $T^c_{x_0}$ into itself by setting $Jv = p'Iq'v$ for any $v \in T^c_{x_0}$. Thus J is of rank n. It also follows that

$$J^2 v = p'Iq'p'Iq'v = p'I^2 q'v = 0$$

for any $v \in T^c_{x_0}$.

Let us put $u = u_H + u_M$ where $u_H \in \boldsymbol{H}^c$, $u_M \in M$, for any $u \in \boldsymbol{G}^c$. We then have $q'v = u_M$ whenever $p'u = v$. Now,

$$\begin{aligned} K_h' Jv &= K_h' p'Iq'v \\ &= p'(\text{adj } h)Iq'v \\ &= p'(\text{adj } h)Iu \end{aligned}$$

where $p'u = v$ since $Iu = I(u_H + u_M) = Iu_M = Iq'v$. In addition we have

$$\begin{aligned} JK_h' v &= p'Iq'K_h' p'u \quad \text{where } p'u = v \\ &= p'Iq'p'(\text{adj } h)u \\ &= p'Iq'p'(w_H + w_M) \quad \text{where } (\text{adj } h)u = w \\ &= p'Iq'p'w_M \\ &= p'Iw_M \\ &= p'I(w_H + w_M) \\ &= p'I(\text{adj } h)u. \end{aligned}$$

Hence for any $v \in T^c_{x_0}$ and any $h \in H$ we have

$$\begin{aligned} (K_h' J - JK_h')v &= p'((\text{adj } h)I - I(\text{adj } h))u \\ &= 0 \end{aligned}$$

where $p'u = v$.

We have thus proven

THEOREM 1. *In order that a homogeneous Lie space $V_{2n} = G/H$ be able to be endowed with a homogeneous almost tangent structure it is* *necessary and sufficient that there exists on* $\mathbf{G}^c$ *a linear operator I satisfying the following conditions*:

(a) *I is zero on* $\boldsymbol{H}^c$,

(b) rank $I = n$,

(c) $I(\mathbf{G}^c) \cap \boldsymbol{H}^c = O$,

(d) $I^2u = 0$ *for any* $u \in \mathbf{G}^c$,

(e) $(\text{adj } h)Iu \equiv I\,(\text{adj } h)u \pmod{\boldsymbol{H}^c}$ *for any* $u \in \mathbf{G}^c$ *and* $h \in H$.

If (7) is satisfied for any $h \in H$, it will in particular be satisfied for the elements $h = \exp(tz)$ of the one parameter subgroup generated by an element z of $\boldsymbol{H}$. By passage to the Lie algebras one deduces from this that the condition (7) leads to

$$[z, Iu] \equiv I[z, u] \pmod{\boldsymbol{H}^c} \tag{8}$$

for any $u \in \mathbf{G}^c$ and $z \in \boldsymbol{H}$. I being a linear operator on $\mathbf{G}^c$, this relation extends to the elements $z \in \boldsymbol{H}^c$.

Conversely, if (8) is satisfied, the condition (7) is satisfied for the elements $h \in H$ such that (adj h) belongs to a one parameter subgroup generated by an element of the Lie algebra of the group (adj H). If this group is connected, (7) is then satisfied for arbitrary h.

THEOREM 2. *In order that a homogeneous Lie space* $V_{2n} = G/H$ *be able to be endowed with a homogeneous almost tangent structure, it is necessary that there exists on* $\mathbf{G}^c$ *a linear operator I satisfying the following conditions*:

(a) *I is zero on* $\boldsymbol{H}^c$,

(b) rank $I = n$,

(c) $I(\mathbf{G}^c) \cap \boldsymbol{H}^c = 0$,

(d) $I^2u = 0$ *for any* $u \in \mathbf{G}^c$,

(e) $[Iu, u'_H] \equiv I[u, u'_H] \pmod{\boldsymbol{H}^c}$ *for any* $u \in \mathbf{G}^c$ *and* $u'_H \in \boldsymbol{H}^c$.

These conditions are sufficient if (adj H) is connected.

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UNIVERSITY OF OTTAWA