ENDOMORPHISM RINGS OF IDEALS IN A COMMUTATIVE REGULAR RING

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Let R be a commutative regular ring, and let R_* be the Boolean ring of idempotents in R. It is well known that the mappings $J \mapsto \hat{J} = J \cap R_*$ and $I \mapsto \bar{I} = RI$ are reciprocal lattice isomorphisms between ideals of R and ideals of R_*. In this paper we make these mappings into functors: To each $\phi \in \mathrm{Hom}_R(J_1, J_2)$, respectively $\Psi \in \mathrm{Hom}_{R_*}(I_1, I_2)$, we associate a homomorphism $\hat{\phi} \in \mathrm{Hom}_{R_*}(\hat{J}_1, \hat{J}_2)$, respectively $\bar{\Psi} \in \mathrm{Hom}_R(\bar{I}_1, \bar{I}_2)$. Using these constructions, we prove that the endomorphism ring of an ideal of R is regular, and that the homological dimension of an ideal is equal to its homological dimension as a module over its endomorphism ring.

For each $x \in R$ let x_* be the (unique) idempotent such that $Rx_* = Rx$. We will use the following relations repeatedly:

$$x_* x = x, \quad (xy)_* = x_* y_*, \qquad \forall x, y \in R. \tag{1}$$

Let ϕ and Ψ be as above, and define $\hat{\phi}$: $\hat{J}_1 \to \hat{J}_2$ and $\bar{\Psi}$: $\bar{I}_1 \to \bar{I}_2$ as follows:

$$\begin{aligned} \hat{\phi}(e) &= \phi(e)_*, & \forall e \in \hat{J}_1, \\ \bar{\Psi}(x) &= x\Psi(x_*), & \forall x \in \bar{I}_1. \end{aligned} \tag{2}$$

Let $\mathcal{I}$, respectively $\mathcal{I}_*$, denote the category of R-ideals and R-homomorphisms, respectively R_*-ideals and R_*-homomorphisms.

THEOREM 1. *The mappings $\phi \mapsto \hat{\phi}$ and $\Psi \mapsto \bar{\Psi}$ are functors. Moreover $\bar{\Psi}\hat{} = \Psi$, so that $\mathcal{I}_*$ is a retract of $\mathcal{I}$.*

Since we are dealing with two distinct additive structures, we use the symbol $\oplus$ to denote addition in the Boolean ring R_*. Then $e_1 \oplus e_2 = e_1 + e_2 - 2e_1e_2$. Often it is convenient to consider the *join* of two idempotents: $e_1 \vee e_2 = e_1 \oplus e_2 \oplus e_1e_2 = e_1 + e_2 - e_1e_2$. Note that $e_1 \oplus e_2 = e_1(1-e_2) \vee e_2(1-e_1)$.

LEMMA 1. *Let I_1 and I_2 be ideals in the Boolean ring $(R_*, \oplus)$, and let h: $I_1 \to I_2$ be any mapping. Consider the following three statements*:

(a) $h(e_1e_2) = e_1h(e_2)$, $\forall e_1 \in R_*$, $e_2 \in I_1$;

(b) $h(e_1e_2) = h(e_1)h(e_2)$, $\forall e_1, e_2 \in I_1$;

(c) $h(e_1 \vee e_2) = h(e_1) \vee h(e_2)$, $\forall e_1, e_2 \in I_1$.

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Then (a) *implies* (b), *and* $h \in \operatorname{Hom}_{R_*}(I_1, I_2)$ *if and only if both* (a) *and* (c) *hold.*

PROOF. Assume (a), and let $e_1, e_2 \in I_1$. Then

$$h(e_1e_2) = h(e_1e_2)h(e_1e_2) = e_1h(e_2)e_2h(e_1) = h(e_1)h(e_2).$$

Next, suppose (a) and (c) are both satisfied. Then

$$\begin{aligned} h(e_1 \oplus e_2) &= (1 - e_2)h(e_1) \vee (1 - e_1)h(e_2) \\ &= (h(e_1) - e_2h(e_1)) \vee (h(e_2) - e_1h(e_2)). \end{aligned}$$

But (a) and (b) together imply that $e_2h(e_1) = e_1h(e_2) = h(e_1)h(e_2)$, and it follows that $h(e_1 \oplus e_2) = h(e_1) \oplus h(e_2)$. Finally, if h is R-linear, then (a) and (b) are valid, and

$$\begin{aligned} h(e_1 \vee e_2) &= h(e_1) \oplus h(e_2) \oplus h(e_1e_2) = h(e_1) \oplus h(e_2) \oplus h(e_1)h(e_2) \\ &= h(e_1) \vee h(e_2). \end{aligned}$$

PROOF OF THEOREM 1. It is easily seen, using (1), that $\hat{\phi}: \hat{J}_1 \to \hat{J}_2$ satisfies condition (a), and hence also condition (b), of Lemma 1. Given $e_1, e_2 \in \hat{J}_1$, set $g = \hat{\phi}(e_1) + \hat{\phi}(e_2) - \hat{\phi}(e_1e_2)$ and $h = \phi(e_1) + \phi(e_2) - \phi(e_1e_2)$. Then $h_* = \hat{\phi}(e_1 \vee e_2)$ and, by condition (b), $g = \hat{\phi}(e_1) \vee \hat{\phi}(e_2)$. Therefore $g^2 = g$, and a direct calculation shows that $hg = h$. If we can show that $g \in Rh$ it will follow that $g = h_*$, and therefore that condition (c) holds. Choose r_1 and r_2 in R such that $r_i\phi(e_i) = \hat{\phi}(e_i)$, $i = 1, 2$, and let $h' = r_1\hat{\phi}(e_1) + r_2\hat{\phi}(e_2) - r_1r_2\phi(e_1e_2)$. Then $h'h = g$, and $g \in Rh$. By Lemma 1, $\hat{\phi} \in \operatorname{Hom}_{R_*}(\hat{J}_1, \hat{J}_2)$. We now show that $\overline{\Psi}$ is R-linear. If $r \in R$ and $x \in \bar{I}_1$, we have $\overline{\Psi}(rx) = rx\Psi(r_*x_*) = rx\Psi(x_*) = r\overline{\Psi}(x)$. Now let $x, y \in \bar{I}_1$. Using the lemma and the fact that Ψ and $\overline{\Psi}$ agree on I_1, we have

$$\begin{aligned} \overline{\Psi}(x + y) &= \overline{\Psi}((x + y)(x_* \vee y_*)) = (x + y)\overline{\Psi}(x_* \vee y_*) \\ &= (x + y)(\overline{\Psi}(x_*) \vee \overline{\Psi}(y_*)) \\ &= (x + y)(\overline{\Psi}(x_*) + \overline{\Psi}(y_*) - \overline{\Psi}(x_*y_*)) = \overline{\Psi}(x) + \overline{\Psi}(y). \end{aligned}$$

The verification that $\hat{\phi}$ and $\overline{\Psi}$ are natural is routine and will be omitted. Clearly $\overline{\Psi}\hat{\ } = \Psi$, and the proof is complete.

The functors in Theorem 1 have the peculiar property of being "exact" but not additive. (Of course $\mathcal{I}$ and $\mathcal{I}_*$ are not exact categories.) For example, let R be a field of characteristic different from 2, and let ϕ and Ψ be the identity maps on R and R_*, respectively. Then $(\phi+\phi)\hat{\ } \neq \hat{\phi} \oplus \hat{\phi}$, and $(\Psi \oplus \Psi)^- \neq \overline{\Psi} + \overline{\Psi}$. This apparent unnaturality is explained by Theorem 2 below.

LEMMA 2. *Let J and K be ideals of the commutative regular ring R, and assume that $K \subseteq J$. Then $\phi K \subseteq K$ for each $\phi \in \mathrm{Hom}_R(J, J)$. In particular, if $S = \mathrm{Hom}_R(J, J)$ then K is a left S-module.*

PROOF. If $x \in K$, $\phi(x) = x\phi(x_*) \in K$.

The following lemma describes explicitly the "image" of the functor $\Psi \mapsto \overline{\Psi}$:

LEMMA 3. *Let $\phi \in \mathrm{Hom}_R(J, J)$. The following conditions are equivalent:* (a) $\phi = \hat{\phi}^-$; (b) $\phi = \overline{\Psi}$ *for some* $\Psi \in \mathrm{Hom}_{R_*}(\hat{J}, \hat{J})$; (c) $\phi^2 = \phi$; (d) $\phi\hat{J} \subseteq \hat{J}$.

PROOF. First observe that the composition in the ring $S = \mathrm{Hom}_R(J, J)$ may be described by

$$(\phi_2\phi_1)(x) = x\phi_2(x_*)\phi_1(x_*). \tag{3}$$

Now (a) obviously implies (b). To show (b) implies (c), we have

$$\overline{\Psi}^2(x) = x\overline{\Psi}(x_*)^2 = x\Psi(x_*) = \overline{\Psi}(x).$$

Next, assuming (c), we have

$$\phi(e)^2 = e\phi(e)^2 = \phi^2(e) = \phi(e),$$

so (d) is satisfied. Finally, if (d) holds

$$\hat{\phi}^-(x) = x\hat{\phi}(x_*) = x\phi(x_*)_* = x\phi(x_*) = \phi(x);$$

therefore (d) implies (a), and the proof is complete.

THEOREM 2. *Let J be an ideal in the commutative regular ring R, let $S = \mathrm{Hom}_R(J, J)$, and let $\hat{S} = \mathrm{Hom}_{R_*}(\hat{J}, \hat{J})$. Then $\hat{S}$ is a Boolean ring, S is a commutative regular ring, and the map $\theta: \Psi \mapsto \overline{\Psi}$ is a ring isomorphism from $\hat{S}$ onto S_*, the Boolean ring of idempotents in S.*

PROOF. By (3), S is commutative. Given $\phi \in S$, let $\phi_* = \hat{\phi}^-$. By Lemma 3, ϕ_* is idempotent, and a simple computation using (3) shows that $\phi_*\phi = \phi$. We will show that $S\phi_* = S\phi$.

For each $x \in R$ there is a unique $\tilde{x} \in R$ satisfying the relations

$$x\tilde{x} = x_*, \qquad x_*\tilde{x} = \tilde{x}. \tag{4}$$

(Choose y such that $xyx = x$ and set $\tilde{x} = yxy$.) Define $\tilde{\phi}: J \to J$ by letting $\tilde{\phi}(x) = x\phi(x_*)^\sim$. Now clearly

$$(xy)^\sim = \tilde{x}\tilde{y}, \qquad \forall x, y \in R; \quad \tilde{e} = e, \qquad \forall e \in R_*. \tag{5}$$

It follows from (1) and (5) that

$$\tilde{\phi}(rx) = r\tilde{\phi}(x), \qquad \forall r \in R, \quad x \in J \tag{6}$$

and

$$\tilde{\phi}(e) = \phi(e)^{\sim}, \qquad \forall e \in \hat{J}. \tag{7}$$

Let $e_1, e_2 \in \hat{J}$ and set $s = \tilde{\phi}(e_1) + \tilde{\phi}(e_2) - \tilde{\phi}(e_1e_2)$. By (6) and (7) we may write $s = e_1\phi(e_1)^{\sim} + e_2\phi(e_2)^{\sim} - e_1e_2\phi(e_1e_2)^{\sim}$. Since $e_i(e_1 \vee e_2) = e_i$, $i = 1, 2$, and $e_1e_2(e_1 \vee e_2) = e_1e_2$, we have

$$\begin{aligned} s\phi(e_1 \vee e_2) &= \phi(e_1)^{\sim}\phi(e_1) + \phi(e_2)^{\sim}\phi(e_2) - \phi(e_1e_2)^{\sim}\phi(e_1e_2) \\ &= \hat{\phi}(e_1) + \hat{\phi}(e_2) - \hat{\phi}(e_1e_2) = \hat{\phi}(e_1) \vee \hat{\phi}(e_2) \\ &= \hat{\phi}(e_1 \vee e_2) = \phi(e_1 \vee e_2)_*. \end{aligned}$$

A similar calculation shows that $s\phi(e_1 \vee e_2)_* = s$. We conclude that $s = \phi(e_1 \vee e_2)^{\sim}$. Taking (7) into account, we have

$$\tilde{\phi}(e_1 \vee e_2) = \tilde{\phi}(e_1) + \tilde{\phi}(e_2) - \tilde{\phi}(e_1e_2). \tag{8}$$

Now if $x, y \in J$,

$$\tilde{\phi}(x + y) = (x + y)\tilde{\phi}(x_* \vee y_*) = \tilde{\phi}(x) + \tilde{\phi}(y), \qquad \text{by (6) and (8).}$$

Therefore $\tilde{\phi} \in \mathrm{Hom}_R(J, J)$. One checks directly that $\phi\tilde{\phi} = \phi_*$ and $\phi_*\tilde{\phi} = \tilde{\phi}$. Hence $S\phi_* = S\phi$, and S is regular.

It remains to be shown that θ is a ring isomorphism. Clearly $\overline{\Psi} = 0$ implies $\Psi = 0$, and θ is onto by Lemma 3. If Ψ_1 and Ψ_2 are in $\hat{S}$ then $(\Psi_2\Psi_1)^- = \overline{\Psi}_2\overline{\Psi}_1$ by Theorem 1. Therefore we need only verify that $(\Psi_1 \oplus \Psi_2)^- = \overline{\Psi}_1 \oplus \overline{\Psi}_2$, (where the symbol $\oplus$ denotes addition in both $\hat{S}$ and S_*). Let $x \in J$. Then

$$\begin{aligned} (\Psi_1 \oplus \Psi_2)^-(x) &= x(\Psi_1 \oplus \Psi_2)(x_*) = x(\Psi_1(x_*) \oplus \Psi_2(x_*)) \\ &= x(\Psi_1(x_*) + \Psi_2(x_*) - 2\Psi_1(x_*)\Psi_2(x_*)) \\ &= x\Psi_1(x_*) + x\Psi_2(x_*) - 2x\overline{\Psi}_1(x_*)\overline{\Psi}_2(x_*) \\ &= \overline{\Psi}_1(x) + \overline{\Psi}_2(x) - 2\overline{\Psi}_1\overline{\Psi}_2(x) \\ &= (\overline{\Psi}_1 + \overline{\Psi}_2 - 2\overline{\Psi}_1\overline{\Psi}_2)(x) = (\overline{\Psi}_1 \oplus \overline{\Psi}_2)(x). \quad \text{Q.E.D.} \end{aligned}$$

REMARK. One can also use Pierce's sheaf-theoretic representation of commutative regular rings [4] to prove that S is regular. It can be shown [6] that S is isomorphic to $\Gamma(U[J], \mathcal{R}(R))$. If

$$\sigma \in \Gamma(U[J], \mathcal{R}(R)),$$

define $\tilde{\sigma}$: $U[J] \to \mathcal{R}(R)$ by letting $\tilde{\sigma}(x) = \sigma(x)^{-1}$ if $\sigma(x) \neq 0$, and $\sigma(x) = 0$ otherwise. Then $\tilde{\sigma}$ is continuous (this requires proof) and $\sigma\tilde{\sigma}\sigma = \sigma$, that is, $\Gamma(U[J], \mathcal{R}(R))$ is regular. Theorem 1 has a similar proof.

We now return to the situation described in Lemma 2 and consider the homological dimension of K as an S-module. Notice that K may be identified with an ideal of S.

THEOREM 3. *Let R, J, K and S be as in Lemma 2. Then* $\text{h.dim}_R(K) = \text{h.dim}_S(K)$.

PROOF. We use a modification of Pierce's projective resolution for ideals in a Boolean ring [5]. Let $\{e_i : i \in I\}$ be a set of idempotent generators for K. If $\delta = (i_0 \cdots i_n) \in I^{n+1}$, let e_δ denote the product $e_{i_0} \cdots e_{i_n}$. For each $n \geqq 0$ let P_n be the R-module of all functions $\phi : I^{n+1} \to R$ such that (i) $\phi(\delta) = 0$ for all but finitely many $\delta \in I^{n+1}$ and (ii) $\phi(\delta) \in Re_\delta$ for each $\delta \in I^{n+1}$. If $\delta \in I^{n+1}$, let $x_\delta \in P_n$ be defined by

$$x_\delta(\delta') = 0, \qquad \delta' \neq \delta,$$
$$x_\delta(\delta') = e_\delta, \qquad \delta' = \delta.$$

Then $P_n = \oplus \sum_{\delta \in I^{n+1}} Rx_\delta$, and, since $Rx_\delta \cong Re_\delta$, P_n is projective. Define $\epsilon : P_0 \to K$ by letting $\epsilon(\phi) = \sum_{i \in I} \phi(i)$. For each $\delta = (i_0 \cdots i_n) \in I^{n+1}$, let $\delta_j = (i_0 \cdots \hat{i}_j \cdots i_n)$, and define $\partial x_\delta = e_\delta \sum_{0 \leq j \leq n} (-1)^j x_{\delta j}$. Clearly ∂ may be extended to a unique R-homomorphism $\partial : P_n \to P_{n-1}$. The sequence

$$P_* : \quad \cdots \to P_n \xrightarrow{\partial} P_{n-1} \to \cdots \to P_1 \xrightarrow{\partial} P_0 \xrightarrow{\epsilon} K \to 0$$

is the desired projective resolution of K. The proof of exactness is very similar to the proof given in [5] and will be omitted.

Given $h \in S$, $\phi \in P_n$ and $\delta \in I^{n+1}$, let $(h\phi)(\delta) = h(\phi(\delta))$. By Lemma 2, $h\phi \in P_n$, so that P_n is an S-module. Now Rx_δ is an S-submodule of P_n; in fact $Rx_\delta \cong Sh_\delta$, where $h_\delta \in S$ is defined by $h_\delta(x) = e_\delta x$. Therefore P_n is a projective S-module. The proof that ∂ and ϵ are S-homomorphisms presents no difficulty since it is enough to check linearity on each direct summand. Therefore P_* is an S-projective resolution of K. For each $n \geqq 0$ let $A_n = \text{im}(\partial : P_{n+1} \to P_n)$. In view of [1, VI., 2.1] the proof will be complete once we verify that A_n is R-projective if and only if it is S-projective.

Suppose A_n is R-projective. Since A_n is generated by $\{\partial x_\delta : \delta \in I^{n+2}\}$ there are idempotents $g_\delta \in R$ such that $A_n = \oplus \sum_{\delta \in I^{n+2}} Rg_\delta \partial x_\delta$, by [5, 3.1]. But $Rg_\delta \partial x_\delta = g_\delta \partial Rx_\delta = g_\delta \partial Sx_\delta = Sg_\delta \partial x_\delta$, so that

$$A_n = \oplus \sum_{\delta \in I^{n+2}} Sg_\delta \partial x_\delta.$$

Now $Sg_\delta \partial x_\delta$ is a cyclic submodule of the projective S-module P_n. Since S is semihereditary $Sg_\delta \partial x_\delta$ must be S-projective [1, I., 6.2]; therefore

A_n is S-projective. Conversely, if A_n is S-projective there are idempotents $f_\delta \in S$ such that $A_n = \oplus \sum_{\delta \in I^{n+2}} Sf_\delta \partial x_\delta$. But $Sf_\delta \partial x_\delta = f_\delta \partial Sx_\delta = f_\delta \partial Rx_\delta = Rf_\delta \partial x_\delta$. As before, it follows that P_n is R-projective. Q.E.D.

If J is any ideal of R and $S = \mathrm{Hom}_R(J, J)$ let $\mu: R \to S$ be the ring homomorphism $x \to \mu_x$, where $\mu_x(y) = xy$ for each $y \in J$. Any S-module acquires an R-module structure via this homomorphism. In view of Theorem 3 one might be led to conjecture that the homological dimension of any S-module is equal to its homological dimension as an R-module. We shall show that in fact either dimension may dominate the other.

Example 1. Let $J = \oplus \sum_{n<\omega} R_n$, where each R_n is a two-element Boolean ring. Embed J as a maximal ideal in the Boolean ring R [**2**, Exercise 1.1]. Then $S = \mathrm{Hom}_R(J, J) \cong \prod_{n<\omega} R_n$. Now R is countable and therefore hereditary. I claim that S is not hereditary. Since S is complete as a Boolean algebra it must be a self-injective ring, by [**4**, 24.2] and [**2**, §21]. But the only self-injective hereditary rings are semisimple with minimum condition, by [**3**] and [**1**, I, 5.4]; hence S cannot be hereditary. If A is any S-module with homological dimension at least 2, we have $\mathrm{h.dim}_S(A) > \mathrm{h.dim}_R(A)$.

For an example of the opposite inequality we shall show that in general S need not be R-projective.

Lemma 4. *Let J be an ideal in the commutative regular ring R and let $e \in R_*$. Then Re is an essential extension of J if and only if* $(0: J) = R(1-e)$.

The proof is straightforward and will be omitted.

Definition. An ideal J of R is *essentially principal* if J is large in some principal ideal. J is *taut* if the homomorphism $\mu: R \to \mathrm{Hom}_R(J, J)$ is an epimorphism.

Theorem 4. *Let J be an ideal in the commutative regular ring R, and let $S = \mathrm{Hom}_R(J, J)$. Then S is projective if and only if J is taut and essentially principal.*

Proof. Consider the sequence

$$\text{(E)} \qquad 0 \to (0: J) \to R \xrightarrow{\mu} S \to 0$$

which is clearly exact, except possibly at S. Suppose S is projective. By [**5**, 3.2], μR is a direct summand of S. But μR is large in S; therefore $\mu R = S$, so that J is taut. Since S is projective, (E) is split exact.

Therefore $(0: J)$ is principal, and, by Lemma 4, J is essentially principal. Conversely, if J is taut and essentially principal, $(0: J)$ is a direct summand of R, and (E) is split exact. Therefore S is projective.

COROLLARY. *Let R be a commutative regular ring. The following conditions are equivalent*: (a) $\mathrm{Hom}_R(J, J)$ *is projective for each ideal* J; (b) *every ideal of R is taut*; (c) *R is self-injective.*

PROOF. (a) implies (b) by Theorem 4. Also (b) and (c) are equivalent, by Lemma 2 and [1, I, 3.2]. Suppose (c) holds. Then the maximal ideal space of R is extremally disconnected [4, 24.1]; hence R_* is complete as a Boolean algebra [2, §21]. Letting $e = \bigvee \hat{J}$, we see that Re is an essential extension of J. By Theorem 4, $\mathrm{Hom}_R(J, J)$ is projective. The proof is now complete.

The ideal J in Example 1 is essentially principal but not taut. The following is an example of an ideal which is taut but not essentially principal.

EXAMPLE 2. Let X_1 and X_2 be disjoint copies of the space of ordinals $\leqq \Omega$ with the order topology. Let X be the space obtained from $X_1 \cup X_2$ by identifying the two copies of the point Ω. Let $U = X_1 - \{\Omega\}$. Let R be the Boolean ring of X and let J be the ideal corresponding to the open set $U \subseteq X$. Then J is taut, since any 2-valued function on U is eventually constant. But J is not essentially principal, since U^- is not open in X.

Much of the work of this paper was motivated by the following question: If two commutative regular rings have homeomorphic maximal ideal spaces, do they necessarily have the same global dimension? The obvious approach is to try to prove that $\mathrm{h.dim}_R(J) = \mathrm{h.dim}_{R_*}(\hat{J})$, whenever J is an ideal of the commutative regular ring R. Theorems 2 and 3 reduce the problem to the special case in which J is large and taut in R. To see this, note that μJ is always large and taut in $S = \mathrm{Hom}_R(J, J)$. Our assertion now follows from

THEOREM 5. *Let J be an ideal in the commutative regular ring R, let $S = \mathrm{Hom}_R(J, J)$, and let $\mu: R \to S$ be the multiplication map. If* $\mathrm{h.dim}_S(\mu J) = \mathrm{h.dim}_{S_*}((\mu J)\hat{})$, *then* $\mathrm{h.dim}_R(J) = \mathrm{h.dim}_{R_*}(\hat{J})$.

PROOF. The map μ defines an S-isomorphism from J onto μJ. Similarly, if $\hat{S} = \mathrm{Hom}_{R_*}(\hat{J}, \hat{J})$, we have an $\hat{S}$-isomorphism ν from $\hat{J}$ onto $\nu\hat{J} \subseteq \hat{S}$. Now it is easily checked that the ring isomorphism of Theorem 2 takes $\nu\hat{J}$ onto $(\mu J)\hat{}$. The desired result now follows by applying Theorem 3 to the rings R and R_*.

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