A NOTE ON THE GLOBAL DIMENSION OF SUBRINGS

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1. **Introduction.** Let R be a ring with identity and e an idempotent of R. Chase [3] shows that if each principal right ideal of R is R-projective (R is a right PP ring), then eRe is a right PP ring. Small [6], shows that if R is right Noetherian and right hereditary, then so is eRe.

In this note it is shown that if R satisfies one of the properties, right PP, right semihereditary, right hereditary or $\mathrm{WGD}(R) \leqq 1$, then eRe satisfies the same property. An example is given for which the right global dimension of R, $rGD(R)$, is 2, but $rGD(eRe)$ is infinite.

2. **Definitions and notation.** R always denotes a ring with identity and all modules are unital. The Jacobson radical of R is denoted $J = J(R)$. A ring R is local if J consists of all the nonunits.

If e is an idempotent of R, then a right R-module P_R is e-free, if it is a direct sum of copies of eR. A right R-module is e-accessible if it is the epimorphic image of an e-free module.

For homological notions the reader is referred to [1].

3. **Results.** In the following $\otimes$ denotes tensor product over R.

THEOREM 1. *Let R be a ring and e an idempotent of R, $S = eRe$, and M_R an e-accessible R-module such that M is R-projective (R-flat), then $(M \otimes Re)_S$ is S-projective (S-flat).*

PROOF. By hypothesis there is an exact sequence of right R-modules

$$0 \to K \to P \to M \to 0 \tag{1}$$

with P e-free, hence projective. Since Re is a flat left R-module the induced sequence of right S-modules

$$0 \to K \otimes \mathrm{Re} \to P \otimes \mathrm{Re} \to M \otimes \mathrm{Re} \to 0 \tag{2}$$

is exact.

If M is R-projective, then (1) is split exact and therefore so is (2). Since $P \otimes Re$ is a free S-module it follows that $M \otimes Re$ is S-projective.

Now suppose that M is R-flat. First, we remark that [2, Proposition 2.2] is valid if F_R is R-projective and not necessarily R-free as may be easily verified. In view of this remark it is sufficient to show that for $v \in K \otimes Re$, there is an S-homomorphism $\phi : P \otimes Re \to K \otimes Re$

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such that $\phi(v)=v$. Clearly, $v=u\otimes e$ for some u in K, so there is an R-homomorphism $\psi: P\to K$ such that $\psi(u)=u$, then the desired map is $\phi=\psi\otimes 1$ and the theorem follows.

COROLLARY 1. *If R is a ring, e an idempotent of R, $S=eRe$, and R satisfies one of the properties, right PP, right semihereditary, right hereditary or* $\mathrm{WGD}(R)\leqq 1$, *then S satisfies the same property.*

PROOF. Let I be a right ideal of S, and IR the right ideal of R generated by I, then clearly IR is e-accessible and finitely generated on the same number of generators as I, if I is finitely generated. Now, $IR\otimes Re$ is S-isomorphic to $IRRe=IeRe=I$ and the corollary follows.

COROLLARY 2. *Let P be a finitely generated projective right R-module and S its R-endomorphism ring. If R is right semihereditary, right hereditary or* $\mathrm{WGD}(R)\leqq 1$, *then S has the same property.*

PROOF. P is a direct summand of a free right R-module F with a finite basis, say n elements. By writing R-endomorphisms of F to the left of elements of F, the R-endomorphism ring of F may be identified with R_n, the ring of all $n\times n$ matrices over R. If R has one of the properties mentioned in the hypothesis of this corollary, then R_n has the same property by [5], or what is the same, by Morita equivalence of the categories of right R-modules and right R_n-modules.

Since P is a direct summand of F, $P=eF$ for some idempotent e of R_n and therefore S is ring isomorphic to eR_ne and this corollary follows from Corollary 1.

An example will show that there is no general theorem: If $\mathrm{rGD}(R)\leqq n$, then $\mathrm{rGD}(eRe)\leqq n$; for $n\geqq 2$. The difficulty arises in the fact that submodules of e-free modules are not necessarily e-accessible, otherwise the obvious techniques would yield such a theorem.

The following example is from [4]. Let K be a field and R the ring of all 3×3 matrices of the form

$$\begin{bmatrix} a & 0 & 0 \\ c & b & 0 \\ d & f & a \end{bmatrix}, \quad \text{where } a, b, c, d, f \text{ are in } K.$$

In [4] it is shown, or it may be easily verified that R is a semiprimary ring with $\mathrm{rGD}(R)=\mathrm{lGD}(R)=2$. Let e be the idempotent

$$e=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

then eRe is the ring of all matrices of the form

$$\begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ f & 0 & a \end{bmatrix}, \quad \text{where } a, f \text{ are in } K.$$

It is easily verified that eRe is a local ring whose Jacobson radical is a simple left and right ideal of eRe, which is isomorphic to the ring modulo its Jacobson radical necessarily, hence is not eRe projective. This fact yields a projective resolution of this simple ideal U of the form (regarded as a left or right eRe module)

$$\cdots eRe \rightarrow eRe \rightarrow eRe \rightarrow U \rightarrow 0,$$

so $\mathrm{rGD}(eRe) = \mathrm{lGD}(eRe)$ is infinite.

References

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