MANY TOPOLOGICAL ABELIAN GROUPS HAVE DENSE DIVISIBLE SUBGROUPS

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1. **Introduction.** If G is a complete, metrizable, topological abelian group such that the groups

$$G^m = \{x \in G \mid my = x \quad \text{for some } y \in G\}$$

are dense in G, then Theorem 2.1 of this paper shows that G has a dense divisible subgroup K. An example appears in §3 of a group G that is not divisible even though G has a dense divisible subgroup. Also some of the possibilities are examined when G is itself divisible. It should be remembered that the set of all divisible elements in an abelian group forms a subgroup.

2. **The existence of dense divisible subgroups.** Let G be a complete, metrizable, topological abelian group. Define $G^m = \{x \in G \mid my = x$ for some $y \in G\}$. Suppose that G^m is dense in G for all $m > 0$.

Let $\{U_n\}_{n=1}^\infty$ be an open basis of the identity of G such that $U_{n+1} + U_{n+1} \subset U_n$. Choose an open neighborhood V_n of the identity of G such that

$$\underbrace{V_n + \cdots + V_n}_{(n!\ \text{times})} \subset U_n.$$

Next a sequence $\{x_i\}_{i=1}^\infty$ will be defined inductively. Given $x \in G$, we suppose x_i, $1 \leq i \leq n$ are given with $x_1 = x$ and $((j+1)x_{j+1} - x_j) \in V_{j+1}$ whenever x_j, x_{j+1} are defined. Define x_{n+1} so that $((n+1)x_{n+1} - x_n) \in V_{n+1}$. We can choose such an x_{n+1} since G^{n+1} is dense in G. For each n consider the sequence $\{y_{n,m}\}_{m=n+1}^\infty$ where

$$y_{n,m;\, m>n} = (m)(m-1) \cdots (n+1)x_m.$$

The claim is that $\{y_{n,m}\}_{m=n+1}^\infty$ is a Cauchy sequence. For consider

$$\begin{aligned} y_{n,m+1} - y_{n,m} &= (m+1)(m) \cdots (n+1)x_{m+1} - (m)(m-1) \cdots (n+1)x_m \\ &= (m)(m-1) \cdots (n+1)[(m+1)x_{m+1} - x_m] \in (m)(m-1) \cdots (n+1) \\ &\qquad V_{m+1} \subset U_{m+1}. \end{aligned}$$

Thus $\{y_{n,m}\}_{m=n+1}^\infty$ forms a Cauchy sequence since $U_{m+1} + U_{m+1} \subset U_m$.

Let y_n be the element which $\{y_{n,m}\}_m$ converges to. The claim is that $(n+1)y_{n+1} = y_n$. Given U_j, we may choose m so that $y_{n+1,m}$ and

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$y_{n,m}$ satisfy $y_{n+1,m}-y_{n+1}\in U_{j+n+2}$ and $y_{n,m}-y_n\in U_{j+1}$. Thus by the definition of $y_{n,m}$ and $y_{n+1,m}$, $(m)(m-1)\cdots(n+2)x_m-y_{n+1}\in U_{j+n+2}$ and $(m)(m-1)\cdots(n+1)x_m-y_n\in U_{j+1}$. Hence

$$(n+1)y_{n+1}-y_n=(n+1)y_{n+1}-(m)(m-1)\cdots(n+1)x_m$$
$$+(m)(m-1)\cdots(n+1)x_m-y_n\in U_{j+1}+U_{j+1}\subset U_j.$$

Since this is true for all j, we have $(n+1)y_{n+1}=y_n$.

As a consequence $n!y_n=y_1$ for all n and y_1 is divisible for all n.

Furthermore,

$$y_1-x=(y_1-y_{1,m})+(y_{1,m}-y_{1,m-1})+(y_{1,m-1}-y_{1,m-2})+\cdots$$
$$+(y_{1,3}-y_{1,2})+(y_{1,2}-x)\in U_1+U_m+U_{m-1}+\cdots$$
$$+U_3+U_2\subset U_1+U_1$$

since for any $j\geqq n$, $y_{n,j+1}-y_{n,j}\in U_{j+1}$, $y_{1,2}=2x_2$, $y_{1,2}-x=2x_2-x_1\in U_2$ and we may choose m so that $y_1-y_{1,m}\in U_1$.

What this shows since U_1 can be chosen as small a neighborhood as desired and since x was arbitrary, is that the divisible elements are dense in G.

THEOREM 2.1. *A complete, metrizable topological abelian group G has a dense divisible subgroup if and only if the G^m are dense in G for all $m>0$.*

PROOF. If the G^m are dense, then it was just shown that G has a dense divisible subgroup. On the other hand if G has a dense divisible subgroup H, $G^m\cap H=H$ is dense in G.

The next theorem gives one example of groups which have a dense divisible subgroup.

THEOREM 2.2. *Let G be a topological abelian group. If G has a dense, cyclic subgroup H and G has no proper open subgroups, then the subgroups G^n are dense for all $n>0$.*

PROOF. Suppose one of the G^n for $n>0$ is not dense in G. Then if $H_1=G^n\cap H$, let H_2 be the closure of H_1 in H. Since H is algebraically isomorphic to the integers and H_2 is not the zero subgroup, H_2 is open in H. Hence since H is dense in G, the closure L of H_2 in G is an open subgroup of G. But since we supposed that G^n is not dense in G, L would be a proper open subgroup of G which is a contradiction.

A special class of groups like those of Theorem 2.1 are those which are divisible as well as being complete and metrizable. If H is a divisible group such that the map $\phi_{1/n}:x\rightarrow(1/n)x$ is well defined and a continuous homomorphism for all $x\in H$, then the map $\phi_{1/n}$ extends to a continuous homomorphism of the completion G of H. Naturally such a G must be torsion free. Any topological vector space is one example of such a group.

Suppose on the other hand G is complete, metrizable, separable, and divisible. Then $\phi_n:G\to G$ is continuous and onto. Thus by the open mapping theorem [1, p. 99], ϕ_n is open.

If G is also torsion free, then the mapping $\phi_{1/n}:x\to(1/n)x$ is well defined and is a homomorphism. Since ϕ_n is open, $\phi_{1/n}=\phi_n^{-1}$ is continuous. The solenoidal group Σ_a, a $=(1, 2, 3, \cdots)$ [2, p. 406 and p. 388] is one example of such a torsion free group G. The a-dic solenoid Σ_a is compact, connected, divisible, and torsion free. But even though the mappings $\phi_{1/n}$ are continuous for all $n>0$, Σ_a is not a topological vector space.

The circle group is an example of a connected, complete, metrizable divisible group such that the mappings $\phi_{1/n}$ are not well defined for any $n>0$.

3. **The size of the maximal divisible subgroup of G.** If G_1 and G_2 are topological groups and $\phi:G_1\to G_2$ is a homomorphism, then ϕ is called almost open [1, p. 14 and p. 89] if for every $x\in G$, $y=\phi(x)$, and neighborhood U of x, the closure of $\phi(U)$, $\mathrm{Cl}(\phi(U))$, is a neighborhood of $\phi(y)$. If $\phi:G_1\to G_2$ is almost open and H_1 is a dense subgroup of G_1, then it can be checked that ϕ restricted to H_1 is almost open.

Suppose that we are given an almost open homomorphism $\phi':H_1\to H_2$. If H_1 is dense in G_1, H_2 is dense in G_2, and $\phi:G_1\to G_2$ is an extension of ϕ' to G_1, then it can be checked that ϕ is almost open.

Even if the G^m are dense in G, the maps $\phi_n:G\to G$ do not have to be almost open. To see this, first define on the additive integers a topology called the group series topology. Let a set $S=\{(p_i, M_i)\}_{i=1}^{\infty}$ of pairs of positive integers be given such that $M_i=2^i$, the greatest common divisor of $\{p_i\}_{i=n}^{\infty}$ is 1 for all n, and such that $3\sum_{i=1}^{j-1} M_i<p_j$ for all j. Define

$$U_n = \left\{x \mid x = \sum_{i=n}^{N} m_i p_i \text{ for } |m_i| \leqq 2^{-n}M_i, \quad \text{for } m_i \text{ and } N \text{ integers}\right\}.$$

These U_n satisfy the axioms [1, p. 46] for a basis of the neighborhoods of the identity for a topological abelian group, i.e., $U_n=-U_n$, and $U_{n+1}+U_{n+1}\subset U_n$. For if $x,\ y\in U_{n+1}$, then $x=\sum_{i=1}^N m_ip_i$ and $y=\sum_{i=1}^N m_i'p_i$ for some integers m_i, m_i', N such that $|m_i|\leqq(1/2^{n+1})M_i$, $|m_i'|\leqq(1/2^{n+1})M_i$. Thus $x+y$ can be written as $x+y=\sum_{i=1}^N (m_i+m_i')p_i$ with $|m_i+m_i'|\leqq(1/2^n)M_i$. This proves that $x+y\in U_n$ and as a result $U_{n+1}+U_{n+1}\subset U_n$.

If $x\in U_n$, then $x=\sum_{i=1}^N m_ip_i$ for some integers N, m_i such that $|m_i|\leqq(1/2^n)M_i$. Thus $-x=\sum_{i=1}^N (-m_i)p_i$ and $|-m_i|=|m_i|$

$\leqq (1/2^n)M_i$. Whence $-x \in U_n$ and therefore $-U_n = U_n$.

Furthermore if $x \in U_1$, then there is a unique set of integers $\{m_i\}_i$ with $x = \sum_i^N m_i p_i$ and $|m_i| \leqq M_i$. For if not for some m_i, m_i', N, and j, $x = \sum_{i=1}^N m_i p_i = \sum_{i=1}^N m_i' p_i$ with $|m_i| \leqq M_i$, $|m_i'| \leqq M_i$ and j the largest index such that $m_j \neq m_j'$. Then $|(m_j - m_j')p_j| = |\sum_{i=1}^{j-1} (m_i - m_i')p_i| \leqq \sum_{i=1}^{j-1} 2M_i$. But since the p_i were chosen so that $p_j > 3\sum_{i=1}^{j-1} M_i$, then we have $|(m_j - m_j')p_j| > 3\sum_{i=1}^{j-1} M_i$ which is a contradiction. Henceforth if we write $x = \sum_i m_i p_i$, it will be assumed that $|m_i| \leqq M_i$. Since the greatest common divisor of the elements of any U_n is 1, then the smallest subgroup of the integers containing any U_n is the integers. Thus if the topological group H is the integers under the topology having a basis of the identity the U_n, H has no proper open subgroups and H^m is dense in H for all $m > 0$.

If $\phi_2: H \to H$ is defined by $\phi_2: x \to 2x$, then the claim is that Cl $(\phi_2(U_3))$ ls not a neighborhood of 0. If $\mathrm{Cl}(\phi_2(U_3))$ is a neighborhood of O, then we can find some U_j with $j \geqq 3$ such that $U_j \subset \mathrm{Cl}(\phi_2(U_3))$.

In particular $p_j \in \mathrm{Cl}(\phi_2(U_3))$ and there is a $c \in \phi_2(U_3)$ such that $p_j - c \in U_{j+1}$. If $c = \sum_{i=1}^N n_i p_i$, all n_i are divisible by 2 since $c \in \phi_2(U_3)$. Also

$$p_j - c \in U_j + \phi_2(U_3) \subset U_2 + U_2 \subset U_1$$

implies that we can write $p_j - c = \sum_{i=1}^{N'} q_i p_i$ with the q_i unique. Since 2 divides n_j, $q_j = 1 - n_j$ is not zero. This contradicts the fact that $p_j - c \in U_{j+1}$. Hence $\mathrm{Cl}(\phi_2(U_3))$ is not a neighborhood of 0 and ϕ_2 is not almost open.

Let G be the completion of H, the integers, under this topology and let K be the maximal divisible group of G. K is dense in G by Theorem 2.1. Suppose $K = G$.

Let $\phi_2': x \to 2x$ be the extension of ϕ_2 to G. Since G is divisible, ϕ_2 is onto. But since G is complete and ϕ_2' is continuous and onto, then ϕ_2' is an open mapping [1, p. 99]. But then since ϕ_2' is almost open, the restriction ϕ_2 of ϕ_2' to H is almost open. This is a contradiction. Hence $K \neq G$.

Thus we have an example of a complete, metrizable group G such that even though the G^n are dense, G is not divisible. The author does not know whether K is always uncountable when G is uncountable.

Bibliography

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