ON A THEOREM OF GOFFMAN AND NEUGEBAUER

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1. Suppose that a function f is defined in an open interval of which $I_0=[a, b]$ is a closed subinterval. In this paper we prove Theorem 2′1 and Theorem 3′2 as two generalizations of the following theorem due to C. Goffman and C. J. Neugebauer [**1**].

THEOREM 1′1. *Suppose that* (i) *f has an approximate derivative f'_{ap} in I_0, and*

(ii) $f'_{\mathrm{ap}}(x)\geqq 0$ *for all x in I_0. Then f is monotone increasing in I_0.*

For definitions and notations used, please see S. Saks [**3**, p. 220]. Unilateral approximate semicontinuity of f is defined in a natural way.

2. THEOREM 2′1. *If* (i) *f is approximately upper semicontinuous* (*a.u.s.c.*) *on the left, and approximately lower semicontinuous* (*a.l.s.c.*) *on the right at each point of I_0, and*

(ii) $\mathrm{Int}\{f(E)\}=\emptyset$, *where* $E=\{x\colon \underline{f}^{+}_{\mathrm{ap}}(x)\leqq 0\}$, *then f is monotone increasing in I_0.*

PROOF. Let there exist two points c, d in I_0, with $c<d$, such that $f(c)>f(d)$. We seek a contradiction. Since $\mathrm{Int}\{f(E)\}=\emptyset$, $f(E)$ does not contain an interval, and, therefore, we can find a number η such that $f(c)>\eta>f(d)$ and

$$\eta\notin f(E). \tag{2′1}$$

We now construct a point ξ in E such that $\eta=f(\xi)$. This will be the desired contradiction. Let $G=\{x\colon f(x)\geqq\eta\}$. Then $c\in G$. Since f is a.l.s.c. on the right at c, it follows that the set $\{x\colon f(x)>\eta\}$, and hence G has right-density unity at c. Therefore, we can find a point x_1 in G, with $x_1>c$, such that

$$\frac{\mu\{G\cap(c, x_1)\}}{x_1-c}\geqq\frac{1}{2}\,. \tag{2′2}$$

Now we adopt a technique due to Goffman and Neugebăer [**1**], and proceed as follows.

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Let $\mathcal{H}$ denote the family of all subsets H of G having the following property: If $x_1, x_2 \in H$, and $x_1 < x_2$, then

$$\frac{\mu\{G \cap (x_1, x_2)\}}{x_2 - x_1} \geqq \frac{1}{2}. \tag{2'3}$$

$\mathcal{H}$ is not empty, since from (2'2) it is evident that $H = \{c, x_1\}$ belongs to $\mathcal{H}$. We now partially order $\mathcal{H}$ by set-inclusion. It is easily verified that every chain in $\mathcal{H}$ has an upper bound in $\mathcal{H}$. By Zorn's Lemma, we conclude that $\mathcal{H}$ has a maximal member H_0, say. Let

$$\xi = \sup\{x : x \in H_0\}. \tag{2'4}$$

If x belongs to H_0, and if $x < \xi$, we shall show that

$$\frac{\mu\{G \cap (x, \xi)\}}{\xi - x} \geqq \frac{1}{2}. \tag{2'5}$$

From (2'4) it follows that we can find a sequence $\{x_n\}$ of points of H_0 and that $x < x_n \leqq \xi$, and $x_n \to \xi_-$. From (2'3) we have

$$\frac{\mu\{G \cap (x, x_n)\}}{x_n - x} \geqq \frac{1}{2}.$$

Taking the limit as $n \to \infty$, we have

$$\frac{\mu\{G \cap (x, \xi)\}}{\xi - x} \geqq \frac{1}{2}.$$

Thus (2'5) has been established. Suppose that $\xi \notin H_0$. Then (2'5) implies that G has no zero left-density at ξ. Since $G = \{x : f(x) \geqq \eta\}$ and since f is a.u.s.c. on the left at ξ, it follows easily that $\eta \leqq f(\xi)$. If $\xi \in H_0$, then $\xi \in G$ and again $\eta \leqq f(\xi)$. Suppose that $\eta < f(\xi)$ then since f is a.l.s.c. on the right at ξ, we conclude that $G = \{x : f(x) \geqq \eta\}$ has right density 1 at ξ. So, we choose $x \in G$ such that $x > \xi$, and

$$\frac{\mu\{G \cap (\xi, x)\}}{x - \xi} \geqq \frac{1}{2}.$$

Since H_0 is maximal w.r.t. the property (2'3), and $x \notin H_0$, there exists $h_0 \in H_0$ such that

$$\frac{\mu\{G \cap (h_0, x)\}}{x - h_0} < \frac{1}{2}. \tag{2'6}$$

Clearly, $h_0 < \xi$. From (2'5) we have

$$\frac{\mu\{G \cap (h_0, \xi)\}}{\xi - h_0} \geqq \frac{1}{2}.$$

Then we have

$$\frac{\mu\{G \cap (h_0, x)\}}{x - h_0} = \frac{\mu\{G \cap (h_0, \xi)\} + \mu\{G \cap (\xi, x)\}}{x - h_0}$$

$$\geqq \frac{\frac{1}{2}(\xi - h_0) + \frac{1}{2}(x - \xi)}{x - h_0} = \frac{1}{2}.$$

This contradicts (2′6), and we conclude that

$$\eta = f(\xi). \tag{2′7}$$

Finally we show that $\xi \in E$. In fact,

$$\underline{f}^{+}_{\mathrm{ap}}(\xi) = \liminf_{x \to \xi+} \mathrm{ap} \frac{f(x) - f(\xi)}{x - \xi}. \tag{2′8}$$

Using (2′7) we observe that

$$\left\{x : x > \xi, \frac{f(x) - f(\xi)}{x - \xi} > 0\right\} \subseteq G. \tag{2′9}$$

Since G has no unit right-density at ξ (by the above argument), neither has the set on r.h.s. of (2′9). Now (2′8) shows immediately that $\underline{f}^{+}_{\mathrm{ap}}(\xi) \leqq 0$, i.e. $\xi \in E$. We have arrived at the final contradiction, and the proof of the theorem is complete.

3. THEOREM 3′1. *Suppose that* (i) f *is a.u.s.c. on the left everywhere in* I_0, *and*

(ii) $\underline{f}^{+}_{\mathrm{ap}}(x) \geqq 0$ *for all* x *in* I_0. *Then* f *is monotone increasing in* I_0.

PROOF. Let x_1 and x_2 be two points in I_0 with $x_1 < x_2$. We shall show that $f(x_2) \geqq f(x_1)$. Without loss of generality we assume that $x_1 = a$ and $x_2 = b$. Let ϵ, with $\epsilon > 0$, be given. Consider the set $G^* = \{x : f(x) - f(a) \geqq -\epsilon(x - a)\}$. Now we construct the point ξ^* with the help of the set G^* exactly in the same way as we construct the point ξ in Theorem 2′1. It is also established as in Theorem 2′1 that G^* has no zero left-density at ξ^*. Since f is a.u.s.c on the left at ξ^*, it follows easily that $\xi^* \in G^*$. Clearly $\xi^* \leqq b$. Suppose that $\xi^* < b$. Since $\underline{f}^{+}_{\mathrm{ap}}(\xi) \geqq 0$, we can find $x \in G^*$, with $x > \xi^*$, such that

$$\frac{\mu\{G^* \cap (\xi^*, x)\}}{x - \xi^*} \geqq \frac{1}{2}.$$

Now we offer a contradiction argument similar to that which has been used to established (2'7) in Theorem 2'1. We thus conclude that $\xi^*=b$. This gives, since $\xi^*\in G^*$, $f(b)-f(a)\geqq -\epsilon(b-a)$. Since ϵ is arbitrary, we have $f(b)\geqq f(a)$. This completes the proof. The following theorem is a generalization of Theorem 3'1.

THEOREM 3'2. *If* (i) f *is a.u.s.c on the left everywhere in* I_0,

(ii) $\underline{f}^{+}_{ap}(x)\geqq 0$ *almost everywhere in* I_0, *and*

(iii) $\underline{f}^{+}_{ap}(x)> -\infty$ *everywhere in* I_0, *then* f *is monotone increasing in* I_0.

PROOF. Let $E=\{x:\underline{f}^{+}_{ap}(x)<0\}$. By hypothesis (ii) $\mu(E)=0$. By a theorem [4, p. 214] there is a continuous increasing function σ in I_0 such that $\sigma'(x)=+\infty$ for $x\in E$. Let ϵ, with $\epsilon>0$, be given. Consider the function ψ defined on I_0 by: $\psi(x)=f(x)+\epsilon\sigma(x)$. Then we have the following:

(i) ψ is a.u.s.c on the left everywhere in I_0, and

(ii) $\underline{\psi}^{+}_{ap}(x)\geqq \underline{f}^{+}_{ap}(x)+\epsilon\underline{\sigma}^{+}_{ap}(x)\geqq 0$ for all x in I_0. Hence, by Theorem 3'1 ψ is monotone increasing in I_0. Since ϵ is arbitrary, we conclude that f is monotone increasing in I_0. The proof is complete.

We wish to point out hypothesis (iii) in Theorem 3'2 is not redundant. The following example illustrates this.

Let

$$
\begin{aligned}
f(x) &= 2x \quad \text{if } 0 \leqq x \leqq 1 \\
&= 1 \quad\;\; \text{if } 1 < x \leqq 2.
\end{aligned}
$$

f satisfies all the conditions of Theorem 3'2 except at $x=1$, where $\underline{f}^{+}_{ap}(x)=-\infty$. f is not monotone increasing in $[0, 2]$.

4. Referring to Theorem 2'1 we want to estimate how large the exceptional set $\{x:\underline{f}^{+}_{ap}(x)\leqq 0\}$ may be without making the theorem false. In this connection we recall the following theorem.

THEOREM 4'1 [2, p. 199]. *Suppose that* f *is a measurable function in* I_0. *Let*

(i) $S=\{x:f'_{ap}(x)$ *exists, and is finite*$\}$,

and

(ii) $T=\{x:$ *all four approximate derivates are infinite at* $x\}$. *Then* $\mu\{I_0\backslash(S\cup T)\}=0$.

As an analogue to this we propose the following theorem.

THEOREM 4'2. *Let f be a measurable function on I_0 such that*
(i) $E_1=\{x: f'_{\text{ap}}(x)$ *exists and* $\neq 0\}$, *and*
(ii) $E_2=\{x$: *all four approximate derivates are infinite at* $x\}$, *then* $\mu[f\{I_0\setminus(E_1\cup E_2)\}]=0$.

We need the following lemmas.

LEMMA 4'1. *Let f be a measurable function, and λ be a real number. Let* $E=\{x: f'_{\text{ap}}(x)=\lambda\}$. *Then*

$$\mu\{f(E)\} \leqq |\lambda| \mu(E).$$

PROOF. By a theorem of S. Saks [3, p. 239], we can write $E=\bigcup_{n=1}^{\infty} E_n$ where f is absolutely continuous on each E_n ($n=1, 2, \cdots$). The sets E_n may be taken to be pairwise disjoint. Since f is absolutely continuous on E_n, it satisfies Lusin's (N)-condition on E_n [3, p. 225], and it is of bounded variation on E_n ($n=1, 2, \cdots$). Therefore, by a lemma of S. Saks [3, p. 221] there is a function g_n which is of bounded variation in I_0 such that $f(x)=g_n(x)$ for $x\in E_n$. Clearly, $g_n'(x)=f'_{\text{ap}}(x)=\lambda$ whenever $x\in E_n\setminus B_n$, where B_n is a subset of E_n with $\mu(B_n)=0$. Since f satisfies Lusin's (N)-condition over E_n, we have

$$\mu\{f(B_n)\} = 0. \tag{4'1}$$

Using a known result [3, p. 227] we have

$$\mu\{f(E_n\setminus B_n)\} = \mu\{g_n(E_n\setminus B_n)\} \leqq \int_{E_n\setminus B_n} |g_n'(x)| \, dx = |\lambda| \mu(E_n\setminus B_n).$$

From (4'1) we deduce $\mu\{f(E_n)\}\leqq|\lambda|\mu(E_n)$. Since $E=\bigcup_{n=1}^{\infty} E_n$, and E_n are pairwise disjoint, we have

$$\mu\{f(E)\} \leqq \sum_{n=1}^{\infty} \mu\{f(E_n)\} \leqq |\lambda| \sum_{n=1}^{\infty} \mu(E_n) = |\lambda| \mu(E).$$

LEMMA 4'2. *Let f be a measurable function on I_0 and let* $E'=\{x$: *at least one approximate derivate is finite at* $x\}$. *Then f satisfies Lusin's (N)-condition on E'.*

This lemma is in S. Saks [3, pp. 290–292].

PROOF OF THEOREM 4'2. Let $E_3=\{x: f'_{\text{ap}}(x)$ exists, and $=0\}$, and $E_4=\{x: f'_{\text{ap}}(x)$ does not exist, and at least one of the four approximate derivates is finite at $x\}$. Clearly $I_0\setminus(E_1\cup E_2)\subseteq E_3\cup E_4$. Hence, (4′2) $$\mu[f\{I_0\backslash(E_1 \cup E_2)] \leqq \mu\{f(E_3)\} + \mu\{f(E_4)\}.$$

By Lemma 4′1

(4′3) $$\mu\{f(E_3)\} = 0.$$

According to Theorem 4′1 $E_4 \subseteq I_0\backslash(S \cup T)$, and hence $\mu(E_4)=0$. Using Lemma 4′2 we have

(4′4) $$\mu\{f(E_4)\} = 0.$$

From (4′2), (4′3) and (4′4) we have $\mu[f\{I_0\backslash(E_1 \cup E_2)\}]=0$.

THEOREM 4′3. *Suppose that f is a measurable function on I_0. Let* (i) *f be a.u.s.c. on the left and a.l.s.c. on the right everywhere in I_0,*

(ii) *$P=\{x: -\infty < f'_{\text{ap}}(x) < 0\}$, and $\mu(P)=0$, and*

(iii) *$Q=\{x$: all four approximate derivates are infinite with $f^{+}_{\text{ap}}(x) = -\infty$, and Q be countable$\}$. Then f is monotone increasing in I_0.*

PROOF. Let $H=\{x: f^{+}_{-\text{ap}}(x) \leqq 0\}$. Then we have

$$\begin{aligned} H\backslash\{I_0\backslash(E_1 \cup E_2)\} &= H \cap (E_1 \cup E_2) \\ &= (H \cap E_1) \cup (H \cap E_2) \subset P \cup Q. \end{aligned}$$

Now

$$\begin{aligned} \mu\{f(H)\} &\leqq \mu[f\{H\backslash(I_0\backslash(E_1 \cup E_2))\}] + \mu[f\{I_0\backslash(E_1 \cup E_2)\}] \\ &= \mu[f\{H\backslash(I_0\backslash(E_1 \cup E_2))\}] \end{aligned}$$

by Theorem 4′2. Thus

(4′5) $$\mu\{f(H)\} \leqq \mu\{f(P)\} + \mu\{f(Q)\}.$$

By hypothesis (ii), $\mu(P)=0$ and by Lemma 4′2

(4′6) $$\mu\{f(P)\} = 0.$$

Let us suppose that $\mu\{f(Q)\}>0$. Then the cardinality of Q must be equal to that of the continuum, and hence Q is uncountable. This contradicts hypothesis (iii). Hence we conclude that

(4′7) $$\mu\{f(Q)\} = 0.$$

From (4′5), (4′6) and (4′7) we obtain $\mu\{f(H)\}=0$. Therefore $\text{Int}\{f(H)\}=\varnothing$. An application of Theorem 2′1 now completes the proof.

References

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