ON A THEOREM OF GOFFMAN AND NEUGEBAUER

A. P. BAINSAB

1. Suppose that a function $f$ is defined in an open interval of which $I_0 = [a, b]$ is a closed subinterval. In this paper we prove Theorem 2'1 and Theorem 3'2 as two generalizations of the following theorem due to C. Goffman and C. J. Neugebauer [1].

**Theorem 1'1.** Suppose that (i) $f$ has an approximate derivative $f'_a$ in $I_0$, and
(ii) $f'_a(x) \geq 0$ for all $x$ in $I_0$. Then $f$ is monotone increasing in $I_0$.

For definitions and notations used, please see S. Saks [3, p. 220]. Unilateral approximate semicontinuity of $f$ is defined in a natural way.

2. **Theorem 2'1.** If (i) $f$ is approximately upper semicontinuous (a.u.s.c.) on the left, and approximately lower semicontinuous (a.l.s.c.) on the right at each point of $I_0$, and
(ii) $\text{Int}\{f(E)\} = \emptyset$, where $E = \{x: f'_a(x) \leq 0\}$, then $f$ is monotone increasing in $I_0$.

**Proof.** Let there exist two points $c, d$ in $I_0$, with $c < d$, such that $f(c) > f(d)$. We seek a contradiction. Since $\text{Int}\{f(E)\} = \emptyset$, $f(E)$ does not contain an interval, and, therefore, we can find a number $\eta$ such that $f(c) > \eta > f(d)$ and

\[
\eta \in f(E).
\]

We now construct a point $\xi$ in $E$ such that $\eta = f(\xi)$. This will be the desired contradiction. Let $G = \{x: f(x) \geq \eta\}$. Then $c \in G$. Since $f$ is a.l.s.c. on the right at $c$, it follows that the set $\{x: f(x) > \eta\}$, and hence $G$ has right-density unity at $c$. Therefore, we can find a point $x_1$ in $G$, with $x_1 > c$, such that

\[
\frac{\mu\{G \cap (c, x_1]\}}{x_1 - c} \geq \frac{1}{2}.
\]

Now we adopt a technique due to Goffman and Neugebäer [1], and proceed as follows.

Received by the editors March 16, 1969.
Let \( \mathcal{C} \) denote the family of all subsets \( H \) of \( G \) having the following property: If \( x_1, x_2 \subseteq H \), and \( x_1 < x_2 \), then
\[
\frac{\mu \{ G \cap (x_1, x_2) \}}{x_2 - x_1} \geq \frac{1}{2}.
\]

\( \mathcal{C} \) is not empty, since from (2'2) it is evident that \( H = \{ c, x_1 \} \) belongs to \( \mathcal{C} \). We now partially order \( \mathcal{C} \) by set-inclusion. It is easily verified that every chain in \( \mathcal{C} \) has an upper bound in \( \mathcal{C} \). By Zorn’s Lemma, we conclude that \( \mathcal{C} \) has a maximal member \( H_0 \), say. Let
\[
(2'4) \quad \xi = \sup \{ x : x \subseteq H_0 \}.
\]

If \( x \) belongs to \( H_0 \), and if \( x < \xi \), we shall show that
\[
(2'5) \quad \frac{\mu \{ G \cap (x, \xi) \}}{\xi - x} \geq \frac{1}{2}.
\]

From (2'4) it follows that we can find a sequence \( \{ x_n \} \) of points of \( H_0 \) and that \( x < x_n \leq \xi \), and \( x_n \rightarrow \xi \). From (2'3) we have
\[
\frac{\mu \{ G \cap (x, x_n) \}}{x_n - x} \geq \frac{1}{2}.
\]

Taking the limit as \( n \rightarrow \infty \), we have
\[
\frac{\mu \{ G \cap (x, \xi) \}}{\xi - x} \geq \frac{1}{2}.
\]

Thus (2'5) has been established. Suppose that \( \xi \in H_0 \). Then (2'5) implies that \( G \) has no zero left-density at \( \xi \). Since \( G = \{ x : f(x) \geq \eta \} \) and since \( f \) is a.u.s.c. on the left at \( \xi \), it follows easily that \( \eta \leq f(\xi) \). If \( \xi \in H_0 \), then \( \xi \in G \) and again \( \eta \leq f(\xi) \). Suppose that \( \eta < f(\xi) \) then since \( f \) is a.l.s.c. on the right at \( \xi \), we conclude that \( G = \{ x : f(x) \geq \eta \} \) has right density 1 at \( \xi \). So, we choose \( x \in G \) such that \( x > \xi \), and
\[
\frac{\mu \{ G \cap (\xi, x) \}}{x - \xi} \geq \frac{1}{2}.
\]

Since \( H_0 \) is maximal w.r.t. the property (2'3), and \( x \in H_0 \), there exists \( h_0 \in H_0 \) such that
\[
(2'6) \quad \frac{\mu \{ G \cap (h_0, x) \}}{x - h_0} < \frac{1}{2}.
\]

Clearly, \( h_0 < \xi \). From (2'5) we have
\[ \mu \left\{ G \cap (h_0, \xi) \right\} = \frac{1}{2}. \]

Then we have
\[
\frac{\mu \left\{ G \cap (h_0, x) \right\}}{x - h_0} = \frac{\mu \left\{ G \cap (h_0, \xi) \right\} + \mu \left\{ G \cap (\xi, x) \right\}}{x - h_0} \geq \frac{1}{2}.
\]

This contradicts (2'6), and we conclude that
\[ (2'7) \quad \eta = f(\xi). \]

Finally we show that \( \xi \in E \). In fact,
\[ (2'8) \quad f_{\text{ap}}^+(\xi) = \liminf_{x \to \xi^+} \frac{f(x) - f(\xi)}{x - \xi}. \]

Using (2'7) we observe that
\[ (2'9) \quad \left\{ x : x > \xi, \frac{f(x) - f(\xi)}{x - \xi} > 0 \right\} \subseteq G. \]

Since \( G \) has no unit right-density at \( \xi \) (by the above argument), neither has the set on r.h.s. of (2'9). Now (2'8) shows immediately that \( f_{\text{ap}}^+(\xi) \leq 0 \), i.e. \( \xi \in E \). We have arrived at the final contradiction, and the proof of the theorem is complete.

3. Theorem 3'1. Suppose that (i) \( f \) is a.u.s.c. on the left everywhere in \( I_0 \), and
(ii) \( f_{\text{ap}}^+(x) \geq 0 \) for all \( x \) in \( I_0 \). Then \( f \) is monotone increasing in \( I_0 \).

Proof. Let \( x_1 \) and \( x_2 \) be two points in \( I_0 \) with \( x_1 < x_2 \). We shall show that \( f(x_2) \geq f(x_1) \). Without loss of generality we assume that \( x_1 = a \) and \( x_2 = b \). Let \( \epsilon \), with \( \epsilon > 0 \), be given. Consider the set \( G^* = \left\{ x : f(x) - f(a) \geq -\epsilon(x - a) \right\} \). Now we construct the point \( \xi^* \) with the help of the set \( G^* \) exactly in the same way as we construct the point \( \xi \) in Theorem 2'1. It is also established as in Theorem 2'1 that \( G^* \) has no zero left-density at \( \xi^* \). Since \( f \) is a.u.s.c on the left at \( \xi^* \), it follows easily that \( \xi^* \in G^* \). Clearly \( \xi^* \leq b \). Suppose that \( \xi^* < b \). Since \( f_{\text{ap}}^+(\xi) \geq 0 \), we can find \( x \in G^* \), with \( x > \xi^* \), such that
\[ \mu \left\{ G^* \cap (\xi^*, x) \right\} = \frac{1}{2}. \]
Now we offer a contradiction argument similar to that which has been used to established (2'7) in Theorem 2'1. We thus conclude that $\xi^* = b$. This gives, since $\xi^* \in G^*$, $f(b) - f(a) \geq -\epsilon(b - a)$. Since $\epsilon$ is arbitrary, we have $f(b) \geq f(a)$. This completes the proof. The following theorem is a generalization of Theorem 3'1.

**Theorem 3'2.** If (i) $f$ is a.u.s.c on the left everywhere in $I_0$,
(ii) $\int f^+(x) \geq 0$ almost everywhere in $I_0$, and
(iii) $f^+(x) > -\infty$ everywhere in $I_0$, then $f$ is monotone increasing in $I_0$.

**Proof.** Let $E = \{x: f^+(x) < 0\}$. By hypothesis (ii) $\mu(E) = 0$. By a theorem [4, p. 214] there is a continuous increasing function $\sigma$ in $I_0$ such that $\sigma'(x) = +\infty$ for $x \in E$. Let $\epsilon$, with $\epsilon > 0$, be given. Consider the function $\psi$ defined on $I_0$ by $\psi(x) = f(x) + \epsilon \sigma(x)$. Then we have the following:

(i) $\psi$ is a.u.s.c on the left everywhere in $I_0$,
(ii) $\int \psi^+(x) \geq f^+(x) + \epsilon \sigma^+(x) \geq 0$ for all $x$ in $I_0$. Hence, by Theorem 3'1 $\psi$ is monotone increasing in $I_0$. Since $\epsilon$ is arbitrary, we conclude that $f$ is monotone increasing in $I_0$. The proof is complete.

We wish to point out hypothesis (iii) in Theorem 3'2 is not redundant. The following example illustrates this.

Let $$f(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } 1 < x \leq 2. \end{cases}$$

$f$ satisfies all the conditions of Theorem 3'2 except at $x = 1$, where $f^+(x) = -\infty$. $f$ is not monotone increasing in $[0, 2]$.

4. Referring to Theorem 2'1 we want to estimate how large the exceptional set $\{x: f^+(x) \leq 0\}$ may be without making the theorem false. In this connection we recall the following theorem.

**Theorem 4'1** [2, p. 199]. Suppose that $f$ is a measurable function in $I_0$. Let

(i) $S = \{x: f^+(x) \text{ exists, and is finite}\}$,
and

(ii) $T = \{x: \text{all four approximate derivates are infinite at } x\}$. Then $\mu \{I_0 \setminus (S \cup T)\} = 0$.

As an analogue to this we propose the following theorem.
Theorem 4.2. Let $f$ be a measurable function on $I_0$ such that
(i) $E_1 = \{ x : f'_{ap}(x) \exists \text{ and } \neq 0 \}$, and
(ii) $E_2 = \{ x : \text{all four approximate derivates are infinite at } x \}$, then
$\mu \left[ f \left( I_0 \setminus (E_1 \cup E_2) \right) \right] = 0$.

We need the following lemmas.

Lemma 4.1. Let $f$ be a measurable function, and $\lambda$ be a real number. Let $E = \{ x : f'_{ap}(x) = \lambda \}$. Then

$$\mu \{ f(E) \} \leq |\lambda| \mu(E).$$

Proof. By a theorem of S. Saks [3, p. 239], we can write $E = \bigcup_{n=1}^{\infty} E_n$ where $f$ is absolutely continuous on each $E_n$ ($n = 1, 2, \ldots$). The sets $E_n$ may be taken to be pairwise disjoint. Since $f$ is absolutely continuous on $E_n$, it satisfies Lusin's $(N)$-condition on $E_n$ [3, p. 225], and it is of bounded variation on $E_n$ ($n = 1, 2, \ldots$). Therefore, by a lemma of S. Saks [3, p. 221] there is a function $g_n$ which is of bounded variation in $I_0$ such that $f(x) = g_n(x)$ for $x \in E_n$. Clearly, $g_n'(x) = f'_{ap}(x) = \lambda$ whenever $x \in E_n \setminus B_n$, where $B_n$ is a subset of $E_n$ with $\mu(B_n) = 0$. Since $f$ satisfies Lusin's $(N)$-condition over $E_n$, we have

$$\mu \{ f(B_n) \} = 0. \quad (4.1)$$

Using a known result [3, p. 227] we have

$$\mu \{ f(E_n \setminus B_n) \} = \mu \{ g_n(E_n \setminus B_n) \} \leq \int_{E_n \setminus B_n} |g_n'(x)| \, dx = |\lambda| \mu(E_n \setminus B_n).$$

From (4.1) we deduce $\mu \{ f(E_n) \} \leq |\lambda| \mu(E_n)$. Since $E = \bigcup_{n=1}^{\infty} E_n$, and $E_n$ are pairwise disjoint, we have

$$\mu \{ f(E) \} \leq \sum_{n=1}^{\infty} \mu \{ f(E_n) \} \leq |\lambda| \sum_{n=1}^{\infty} \mu(E_n) = |\lambda| \mu(E).$$

Lemma 4.2. Let $f$ be a measurable function on $I_0$ and let $E' = \{ x : \text{at least one approximate derivate is finite at } x \}$. Then $f$ satisfies Lusin's $(N)$-condition on $E'$.

This lemma is in S. Saks [3, pp. 290–292].

Proof of Theorem 4.2. Let $E_3 = \{ x : f'_{ap}(x) \exists \text{ and } = 0 \}$, and $E_4 = \{ x : f'_{ip}(x) \text{ does not exist, and at least one of the four approximate derivates is finite at } x \}$. Clearly $I_0 \setminus (E_1 \cup E_2) \subseteq E_3 \cup E_4$. Hence,
\( (4'2) \) \[ \mu\{f(I_0 \setminus (E_1 \cup E_2))\} \leq \mu\{f(E_3)\} + \mu\{f(E_4)\}. \]

By Lemma 4'1

\( (4'3) \) \[ \mu\{f(E_3)\} = 0. \]

According to Theorem 4'1 \( E_4 \subseteq I_0 \setminus (S \cup T) \), and hence \( \mu(E_4) = 0 \).

Using Lemma 4'2 we have

\( (4'4) \) \[ \mu\{f(E_4)\} = 0. \]

From \((4'2), (4'3)\) and \( (4'4) \) we have \( \mu\{f(I_0 \setminus (E_1 \cup E_2))\} = 0 \).

**Theorem 4'3.** Suppose that \( f \) is a measurable function on \( I_0 \). Let

(i) \( f \) be a.u.s.c. on the left and a.l.s.c. on the right everywhere in \( I_0 \),
(ii) \( P = \{ x: -\infty < f_{ap}(x) < 0 \} \), and \( \mu(P) = 0 \), and
(iii) \( Q = \{ x: \text{all four approximate derivates are infinite with } f_{ap}(x) = -\infty \}, \text{and } Q \text{ be countable} \). Then \( f \) is monotone increasing in \( I_0 \).

**Proof.** Let \( H = \{ x: f_{ap}(x) \leq 0 \} \). Then we have

\[ H \setminus \{ I_0 \setminus (E_1 \cup E_2) \} = H \cap (E_1 \cup E_2) = (H \cap E_1) \cup (H \cap E_2) \subseteq P \cup Q. \]

Now

\[ \mu\{f(H)\} \leq \mu\{f(H \setminus (I_0 \setminus (E_1 \cup E_2)))\} + \mu\{f(I_0 \setminus (E_1 \cup E_2))\} \]

by Theorem 4'2. Thus

\( (4'5) \) \[ \mu\{f(H)\} \leq \mu\{f(P)\} + \mu\{f(Q)\}. \]

By hypothesis (ii), \( \mu(P) = 0 \) and by Lemma 4'2

\( (4'6) \) \[ \mu\{f(P)\} = 0. \]

Let us suppose that \( \mu\{f(Q)\} > 0 \). Then the cardinality of \( Q \) must be equal to that of the continuum, and hence \( Q \) is uncountable. This contradicts hypothesis (iii). Hence we conclude that

\( (4'7) \) \[ \mu\{f(Q)\} = 0. \]

From \( (4'5), (4'6) \) and \( (4'7) \) we obtain \( \mu\{f(H)\} = 0 \). Therefore \( \text{Int}\{f(H)\} = \emptyset \). An application of Theorem 2'1 now completes the proof.
References


Burdwan University, India