ON WU'S FORM OF THE FIRST MAIN THEOREM OF VALUE DISTRIBUTION

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In the First Main Theorem of value distribution for holomorphic maps into a compact Kähler manifold A, the central object is a differential form λ_a on A with a singularity at $a \in A$. The most satisfactory λ_a was given by Wu in [5]. His construction, using the Green function, has the advantage that λ_a is automatically invariant under the group of isometries of A; more precisely, if $i: A \to A$ is an isometry with $i(a) = b$, then $i^*\lambda_b = \lambda_a$. This property is difficult to obtain with any other construction.

In this note we apply the technical machinery developed by Stoll and the author [1], [2], [3], [4] to Wu's λ_a to show that it is suitable for value distribution theory in arbitrary fiber dimension. We also show that the terms of the First Main Theorem are continuous functions.

ν_f will denote the multiplicity of the holomorphic map f; see [2]. O_k will denote any differential form of degree k whose coefficients are locally bounded; the subscript will usually be omitted. Of course it will never mean the same thing twice. On $\mathbf{C}^n$ set $v_1 = (i/2) \sum_{\mu=1}^{n} dz_\mu \wedge d\bar{z}_\mu$ and $v_1^k = v^k/k!$.

If M is a complex manifold and H an open subset of M, then $S = \overline{H} - H$ is a *boundary manifold* of H, provided that for every point of S there is a C^∞ coordinate system $(x_1, \cdots, x_{2n})$ valid in a neighborhood U of the point, such that in U, S is described by $x_1 = 0$ and H by $x_1 < 0$. S will be considered as oriented to the exterior of H.

1. We collect the technical results which we will need.

LEMMA 1. *Let $k \geqq 0$, $0 \leqq s \leqq n$ and $0 \leqq t \leqq n$ be integers such that $s + t < 2n$. Let $\phi(w, z)$ be a differential form of bidegree (s, t) in $w \in \mathbf{C}^n$, dependent on a parameter $z \in \mathbf{C}^n$, whose coefficients are measurable and locally bounded. Let M be a complex manifold, and let $f: M \to \mathbf{C}^n$ be an open holomorphic map. Let K be a compact subset of M. Let $\chi(w, z)$ be a differential form in $w \in M$ dependent also on a parameter $z \in \mathbf{C}^n$, whose coefficients are measurable and locally bounded; then*

$$F(z) = \int_{w \in K} [\log(1/|f(w) - z|)]^k |f(w) - z|^{-(s+t)} \cdot f^*\phi(w, z) \wedge \chi(w, z)$$

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exists for each $z \in \mathbf{C}^n$, *and is a measurable and locally bounded function on* $\mathbf{C}^n$. *If moreover* ϕ *is continuous off the diagonal and* χ *is continuous off* $\{(w, z) \mid f(w) = z\}$, *then* F *is continuous.*

This is a restatement of Lemma 2.6 of [1].

Choose a function h of class C^∞ on $\boldsymbol{R}$ such that $0 \leqq h(x) \leqq 1$ for all x, $h(x) = 0$ if $x \leqq 0$, and $h(x) = 1$ if $x \geqq 1$. A constant $B > 0$ exists such that $2|h'(x)| \leqq B$ for all x.
For $\rho > 0$ define

$$g_\rho(X) = h((2X - \rho)/\rho).$$

Then

(a) $0 \leqq g_\rho(X) \leqq 1$,
(b) $X|g_\rho'(X)| \leqq B$,
(c) $X \leqq \rho/2 \Rightarrow g_\rho(X) = 0$,
(d) $X \geqq \rho \Rightarrow g_\rho(X) = 1$.

Now let $f\colon M \to \mathbf{C}^n$ be a holomorphic map. Define γ_ρ on M by $\gamma_\rho(w) = g_\rho(|f(w)|)$. Then $d\gamma_\rho = g_\rho'(|f|)((df|f) + (f|df))/2|f|$. Set $h_\rho = |f| g_\rho'(|f|)$ and $\eta = ((df|f) + (f|df))/2|f|$. Then the functions h_ρ are uniformly bounded, η is a locally bounded form, and

$$d\gamma_\rho = h_\rho \eta / |f|.$$

Define $L_\rho = \{w \in K \mid \rho/2 \leqq \gamma_\rho(w) \leqq \rho\}$.

LEMMA 2. *Hypotheses as in Lemma* 1, *let also a measurable function* h_ρ *be given for each* $\rho > 0$, *and suppose that these functions are uniformly bounded on* K. *Then*

$$\lim_{\rho \to 0} \int_{L_\rho} h_\rho [\log(1/|f|)]^k |f|^{-(s+t)} f^* \phi \wedge \chi = 0.$$

See Stoll [4, Lemma 1.7].

LEMMA 3. *Hypotheses as above, let* ϕ *be a locally bounded form of degree* $2n-1$ *on* $\mathbf{C}^n$. *Then*

$$\lim_{\rho \to 0} \int_K d\gamma_\rho \wedge |f|^{2-2n} [\log(1/|f|)]^k f^* \phi \wedge \chi = 0.$$

PROOF. The expression under the integral sign is

$$-(h_\rho/|f|)|f|^{2-2n}[\log(1/|f|)]^k f^* \phi \wedge (\eta \wedge \chi).$$

Now decompose ϕ into forms of bidegrees $(n, n-1)$ and $(n-1, n)$ and apply Lemma 2 twice. q.e.d.

LEMMA 4. *Hypotheses as above, and suppose also that H is an open subset of M such that $\overline{H}=K$ and that such that $S=\overline{H}-H$ is either empty or a smooth boundary manifold of H. Let T be the support of χ on S and suppose that $T\cap f^{-1}(0)$ is of measure zero on $f^{-1}(0)$. Then*

$$\lim_{\rho\to 0}\int_H d\gamma_\rho \wedge d^c \log(1/|f|) \wedge |f|^{2-2n} f^* v_n \wedge \chi$$
$$= -[2\pi^n/(n-1)!]\int_{H\cap f^{-1}(0)} \nu_f \chi.$$

See Stoll [4, Lemma 1.8].

2. Let A be a compact Kähler manifold of dimension n. Denote its fundamental form by ω_1, and set $\omega_k=\omega_1^k/k!$, $\omega=\omega_n$. We assume that the metric is normalized so that $\int_A \omega=1$. Denote by $r(z, w)$ the distance between z and w.

Denote by $g(z, w)$ the Green function on $A\times A$. Then

$$\begin{aligned} g(z, w) &= \tfrac{1}{4}(n-2)!\pi^{-n} r(z, w)^{2-2n} + r(z, w)^{3-2n} O(z, w), \quad n>1, \\ g(z, w) &= (1/2\pi)\log(1/r(z, w)) + O(z, w), \quad n=1. \end{aligned} \tag{1}$$

There exists a constant C such that $g(z, w)+C$ is everywhere positive.

Define

$$\lambda_z(w) = (g(z, w)+C)\omega_{n-1}(w). \tag{2}$$

Then $dd^c\lambda_z=\omega$. (See Wu [5, I and II].)

Let $a\in A$, and let $(z_1, \cdots, z_n)$ be a holomorphic coordinate system in a neighborhood U centered at a, such that in the underlying real coordinate system the metric tensor at a is represented by the identity matrix. Set

$$B(z, w) = |z-w|/r(z, w) \tag{3}$$

where $|z-w|$ is computed in the $(z_1, \cdots z_n)$ coordinate system. B is continuous and positive on $U\times U$ and $B(a, a)=1$.

From (1), (2), and (3) we see

$$\begin{aligned} \lambda_z(w) &= |z-w|^{2-2n} O(z, w) v_{n-1}, \quad n>1, \\ \lambda_z(w) &= (1/2\pi)\log(1/|z-w|) + O(z, w), \quad n=1. \end{aligned} \tag{4}$$

From (1), (see Wu [5, II, §8] for justification)

$$
\begin{aligned}
d_w^c\lambda_z(w) &= [-[(n-1)!/2\pi^n]r(z,w)^{1-2n}d_w^c r(z,w) \\
&\quad + O_1(z,w)r(z,w)^{2-2n}] \wedge \omega_{n-1}(w) \quad (\text{if } n > 1) \\
&= [[(n-1)!/2\pi^n]r(z,w)^{2-2n}d_w^c \log(1/r(z,w)) \\
&\quad + O_1(z,w)r(z,w)^{2-2n}] \wedge \omega_{n-1}(w) \qquad (\text{any } n).
\end{aligned} \tag{5}
$$

Now $\log(1/r(z,w)) = \log B(z,w) + \log(1/|z-w|)$ and $\log B(z,w)$ is locally bounded; hence

$$
\begin{aligned}
d_w^c\lambda_z(w) &= [[(n-1)!/2\pi^n]B(z,w)^{2n-2}r(z,w)^{2-2n}d_w^c \log(1/|z-w|) \\
&\quad + O_1(z,w)|z-w|^{2-2n}] \wedge \omega_{w-1}(w).
\end{aligned} \tag{6}
$$

Since $d^c \log(1/|z-w|) = -d^c|z-w|/|z-w|$, we have

$$
d^c\lambda_z(w) = O_{2n-1}|z-w|^{1-2n}. \tag{7}
$$

Finally, look at $z=0$, i.e., at a:

$$
d^c \log(1/|w|) \wedge (\omega_{n-1}(w) - v_{n-1}(w)) = -d^c|w| \wedge (\omega_{n-1}(w) - v_{n-1}(w))/|w|.
$$

Since $\omega_{n-1}(a) = v_{n-1}(a)$, this is bounded. Thus

$$
\begin{aligned}
d^c\lambda_a(w) &= [(n-1)!/2\pi^n]B(a,w)^{2n-2}|w|^{2-2n}d^c \log(1/|w|) \wedge v_{n-1} \\
&\quad + O_{2n-1}(w)|w|^{2-2n}.
\end{aligned} \tag{8}
$$

LEMMA 5. *Let $f: M \to A$ be open and holomorphic, K a compact subset of M, and χ continuous on K; then*

$$
F(z) = \int_K f^*\lambda_z \wedge \chi
$$

exists for each $z \in A$ and is continuous.

PROOF. The question is local on both K and A, so we may assume $f(K) \subset U$ and adopt the above notation. Apply (4) and Lemma 1. q.e.d.

LEMMA 6. *In the same situation,*

$$
G(z) = \int_K f^* d^c\lambda_z \wedge \chi
$$

exists and is continuous on A.

PROOF. Use (7) and Lemma 1.

3. THE RESIDUE THEOREM. *Let M be a complex manifold of dimension $m=n+q$. Let A be a compact Kähler manifold, and let $f: M\to A$ be an open holomorphic map. Let H be an open subset of M. Let $S=\overline{H}-H$ be empty or a boundary manifold of H, and suppose that $\overline{H}$ is compact. Let ψ be a continuous function on $\overline{H}$ satisfying a Lipschitz condition. (Then $d\psi$ exists almost everywhere.) Let χ be a differential form of bidegree (q, q) and class C^1 on M.*

Then $d^cf^*\lambda_a\wedge d\psi\wedge\chi$ and $d^cf^*\lambda_a\wedge\psi\wedge d\chi$ are integrable over H for each $a\in A$.

Also let T be the support of $\psi\chi$ on S, and assume that $T\cap f^{-1}(a)$ is a set of measure zero on $f^{-1}(a)$. Suppose that $d^cf^*\lambda_a\wedge\psi\chi$ is integrable on S. Then

$$\int_S d^cf^*\lambda_a\wedge\psi\chi = -\int_H d^cf^*\lambda_a\wedge d\psi\wedge\chi - \int_H d^cf^*\lambda_a\wedge\psi\wedge d\chi + \int_H f^*\omega\wedge\psi\chi - \int_{H\cap f^{-1}(a)} \nu_f\psi\chi.$$

PROOF. By Stokes' Theorem,

$$\int_S \gamma_\rho d^cf^*\lambda_a\wedge\psi\chi = -\int_H \gamma_\rho d^cf^*\lambda_a\wedge d\psi\wedge\chi - \int_H \gamma_\rho d^cf^*\lambda_a\wedge\psi\wedge d\chi + \int_H \gamma_\rho f^*\omega\wedge\psi\chi + \int_H d\gamma_\rho\wedge d^cf^*\lambda_a\wedge\psi\chi.$$

We let $\rho\to 0$; from Lemma 6 we see that we need only show that the last integral has limit

$$-\int_{H\cap f^{-1}(a)} \nu_f\psi\chi.$$

By (8), it is

$$\int_H d\gamma_\rho\wedge[(n-1)!/2\pi^n]B(a,f)^{2n-2}|f|^{2-2n}\wedge d^c\log(1/|f|)\wedge v_{n-1}\wedge\psi\chi + \int_H d\gamma_\rho f^*O_{2n-1}|f|^{2-2n}\wedge\psi\chi.$$

By Lemma 3, the second term has limit zero. By Lemma 4, the first term has limit

$$-\int_{H\cap f^{-1}(a)} \nu_f\psi\chi B(a, a)^{2n-2}$$

but $B(a, a)=1$. q.e.d.

SINGULAR STOKES THEOREM. *Let M, A, f, H, S, and χ be as above. Let ψ be a function of class C^2 on $\overline{H}$. Then $d^c f^*\lambda_a \wedge d\psi \wedge \chi$, $f^*\lambda_a \wedge dd^c\psi \wedge \chi$, and $f^*\lambda_a \wedge d^c\psi \wedge d\chi$ are integrable over H.*

Assume further that $f^*\lambda_a \wedge d^c\psi \wedge \chi$ is integrable over S. Then

$$\int_S f^*\lambda_a \wedge d^c\psi \wedge \chi = -\int_H d^c f^*\lambda_a \wedge d\psi \wedge \chi + \int_H f^*\lambda_a \wedge dd^c\psi \wedge \chi$$
$$- \int_H f^*\lambda_a \wedge d^c\psi \wedge d\chi.$$

PROOF. Similar to that of the Residue Theorem. Apply Lemmas 5 and 6; then consider

$$\int_H d\gamma_\rho \wedge f^*\lambda_a \wedge d^c\psi \wedge \chi.$$

By (4) it is

$$\int_H d\gamma_\rho \wedge O_0 |f|^{2-2n} f^* v_{n-1} \wedge d^c\psi \wedge \chi$$

if $n > 1$ and

$$\int_H d\gamma_\rho \wedge [(1/2\pi)\log(1/|f|) + O_0] \wedge d^c\psi \wedge \chi$$

if $n = 1$. Now apply Lemma 3. q.e.d.

The so-called Unintegrated Form of the First Main Theorem is obtained by setting $\psi = 1$ and choosing χ so that $d\chi = 0$ in the Residue Theorem. We can now state the:

INTEGRATED FIRST MAIN THEOREM. *Let M, A, f and χ be as before, and assume $d\chi = 0$. Let G be an open subset of M with compact closure $\overline{G}$. Assume that $\Gamma = \overline{G} - G$ is a boundary manifold of G. Let g be open with $\bar{g} \subset G$, and assume that $\gamma = \bar{g} - g$ is a boundary manifold of g. Let ψ be a continuous, real-valued function on M, of class C^2 on $\overline{G} - g$, such that $\psi | g = R$ a constant, $\psi | M - G = 0$, and $0 \leqq \psi(z) \leqq R$ for all $z \in M$. (Such a ψ satisfies a Lipschitz condition on M.) Define*

$$m^\#(a) = -\int_\Gamma f^*\lambda_a \wedge d^c\psi \wedge \chi,$$

$$m^\flat(a) = -\int_\gamma f^*\lambda_a \wedge d^c\psi \wedge \chi,$$

$$N(a) = \int_{f^{-1}(a)} \nu_f \psi \chi,$$

$$T = \int_G f^*\omega \wedge \psi\chi,$$

$$\Delta(a) = \int_G f^*\lambda_a \wedge d^c d\psi \wedge \chi.$$

Then these are continuous functions on A, and

$$N + m^\# - m^\flat = T + \Delta.$$

Note that N, m, $m^\flat$, and T are always nonnegative; but in general we have information on Δ.

This theorem is obtained by applying the Residue Theorem to G, the Singular Stokes Theorem to $G - \bar{g}$, and eliminating the common term; see [**1**] for details. The continuity of Δ is a consequence of Lemma 5, and the continuity of N is a theorem of Stoll [**3**, Theorem 3.8]. Thus $m^\# - m^\flat$ is continuous. Now let h be a C^∞ function with compact support in G and which is 1 in a neighborhood of $\bar{g}$, and replace ψ by $h\psi$. We get another FMT with different (but still continuous) N, T, and Δ. The new $m^\#$ is zero but the new $m^\flat$ is the same as the old. Hence the old $m^\flat$ and $m^\#$ are continuous.

References

1. J. Hirschfelder, *The first main theorem of value distribution in several variables*, Invent. Math. **8** (1969), 1–33.

2. W. Stoll, *The multiplicity of a holomorphic map*, Invent. Math. **2** (1966), 15–58.

3. ———, *The continuity of the fiber integral*, Math. Z. **95** (1967), 87–138.

4. ———, *A general first main theorem of value distribution theory*, Acta Math. **118** (1967), 111–191.

5. H. Wu, *Remarks on the first main theorem in equidistribution theory*. I, II, J. Differential Geometry **2** (1968), 197–202, 369–384.

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