ON WU'S FORM OF THE FIRST MAIN THEOREM OF VALUE DISTRIBUTION

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In the First Main Theorem of value distribution for holomorphic maps into a compact Kähler manifold $A$, the central object is a differential form $\lambda_a$ on $A$ with a singularity at $a \in A$. The most satisfactory $\lambda_a$ was given by Wu in [5]. His construction, using the Green function, has the advantage that $\lambda_a$ is automatically invariant under the group of isometries of $A$; more precisely, if $i: A \to A$ is an isometry with $i(a) = b$, then $i^*\lambda_b = \lambda_a$. This property is difficult to obtain with any other construction.

In this note we apply the technical machinery developed by Stoll and the author [1], [2], [3], [4] to Wu's $\lambda_a$ to show that it is suitable for value distribution theory in arbitrary fiber dimension. We also show that the terms of the First Main Theorem are continuous functions.

$v_f$ will denote the multiplicity of the holomorphic map $f$; see [2]. $O_k$ will denote any differential form of degree $k$ whose coefficients are locally bounded; the subscript will usually be omitted. Of course it will never mean the same thing twice. On $\mathbb{C}^n$ set $v_1 = (i/2) \sum_{\mu=1}^{n} dz_{\mu} \wedge d\bar{z}_{\mu}$ and $v_1^k = v_1^k/k!$.

If $M$ is a complex manifold and $H$ an open subset of $M$, then $S = H - H$ is a boundary manifold of $H$, provided that for every point of $S$ there is a $C^\infty$ coordinate system $(x_1, \ldots, x_{2n})$ valid in a neighborhood $U$ of the point, such that in $U$, $S$ is described by $x_1 = 0$ and $H$ by $x_1 < 0$. $S$ will be considered as oriented to the exterior of $H$.

1. We collect the technical results which we will need.

Lemmma 1. Let $k \geq 0$, $0 \leq s \leq n$ and $0 \leq t \leq n$ be integers such that $s + t < 2n$. Let $\phi(w, z)$ be a differential form of bidegree $(s, t)$ in $w \in \mathbb{C}^n$, dependent on a parameter $z \in \mathbb{C}^n$, whose coefficients are measurable and locally bounded. Let $M$ be a complex manifold, and let $f: M \to \mathbb{C}^n$ be an open holomorphic map. Let $K$ be a compact subset of $M$. Let $\chi(w, z)$ be a differential form in $w \in M$ dependent also on a parameter $z \in \mathbb{C}^n$, whose coefficients are measurable and locally bounded; then

$$F(z) = \int_{w \in K} \left[ \log(1/|f(w) - z|) \right]^k |f(w) - z|^{-(s+t)} \cdot f^*\phi(w, z) \wedge \chi(w, z)$$

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exists for each \( z \in \mathbb{C}^n \), and is a measurable and locally bounded function on \( \mathbb{C}^n \). If moreover \( \phi \) is continuous off the diagonal and \( \chi \) is continuous off \( \{(w, z) \mid f(w) = z\} \), then \( F \) is continuous.

This is a restatement of Lemma 2.6 of [1].

Choose a function \( h \) of class \( C^\infty \) on \( \mathbb{R} \) such that \( 0 \leq h(x) \leq 1 \) for all \( x \), \( h(x) = 0 \) if \( x \leq 0 \), and \( h(x) = 1 \) if \( x \geq 1 \). A constant \( B > 0 \) exists such that \( 2|h'(x)| \leq B \) for all \( x \).

For \( \rho > 0 \) define

\[
g_\rho(X) = h((2X - \rho)/\rho).
\]

Then

(a) \( 0 \leq g_\rho(X) \leq 1 \),

(b) \( X \left| g_\rho'(X) \right| \leq B \),

(c) \( X \leq \rho/2 \Rightarrow g_\rho(X) = 0 \),

(d) \( X \geq \rho \Rightarrow g_\rho(X) = 1 \).

Now let \( f: M \to \mathbb{C}^n \) be a holomorphic map. Define \( \gamma_\rho \) on \( M \) by \( \gamma_\rho(w) = g_\rho(|f(w)|) \). Then \( d\gamma_\rho = g_\rho'(|f|)(df|f| + (df|df|))/2|f| \). Set \( h_\rho = |f| g_\rho'(|f|) \) and \( \eta = ((df|f| + (df|df|))/2|f| \). Then the functions \( h_\rho \)

are uniformly bounded, \( \eta \) is a locally bounded form, and

\[
d\gamma_\rho = h_\rho \eta/|f|.
\]

Define \( L_\rho = \{ w \in K \mid \rho/2 \leq \gamma_\rho(w) \leq \rho \} \).

**Lemma 2.** Hypotheses as in Lemma 1, let also a measurable function \( h_\rho \) be given for each \( \rho > 0 \), and suppose that these functions are uniformly bounded on \( K \). Then

\[
\lim_{\rho \to 0} |K| \int_{L_\rho} h_\rho [\log(1/|f|)]^k |f|^{-(n + t)f} \phi \wedge \chi = 0.
\]

See Stoll [4, Lemma 1.7].

**Lemma 3.** Hypotheses as above, let \( \phi \) be a locally bounded form of degree \( 2n - 1 \) on \( \mathbb{C}^n \). Then

\[
\lim_{\rho \to 0} |K| \int_{K} d\gamma_\rho \wedge |f|^{2 - 2n} [\log(1/|f|)]^k f^{*}\phi \wedge \chi = 0.
\]

**Proof.** The expression under the integral sign is

\[-(h_\rho/|f|) |f|^{2 - 2n} [\log(1/|f|)]^k f^{*}\phi \wedge (\eta \wedge \chi).
\]

Now decompose \( \phi \) into forms of bidegrees \((n, n - 1)\) and \((n - 1, n)\) and apply Lemma 2 twice.  q.e.d.
Lemma 4. Hypotheses as above, and suppose also that $H$ is an open subset of $M$ such that $\overline{H} = K$ and that such that $S = \overline{H} - H$ is either empty or a smooth boundary manifold of $H$. Let $T$ be the support of $\chi$ on $S$ and suppose that $T \cap f^{-1}(0)$ is of measure zero on $f^{-1}(0)$. Then

$$\lim_{\rho \to 0} \int_{\partial T} d\gamma \wedge d\theta \log(1/|f|) \wedge |f|^{2-2n} \nu_n \wedge \chi$$

$$= - \frac{2\pi^n}{(n-1)!} \int_{H \cap f^{-1}(0)} \nu_f \chi.$$

See Stoll [4, Lemma 1.8].

2. Let $A$ be a compact Kähler manifold of dimension $n$. Denote its fundamental form by $\omega$, and set $\omega_k = \omega_k^k/k!$, $\omega = \omega_n$. We assume that the metric is normalized so that $\int_A \omega = 1$. Denote by $r(z, w)$ the distance between $z$ and $w$.

Denote by $g(z, w)$ the Green function on $A \times A$. Then

$$g(z, w) = \frac{1}{4}(n - 2)! \pi^{-n} r(z, w)^{2-2n} + r(z, w)^{3-2n} O(z, w), \quad n > 1,$$

(1)

$$g(z, w) = \frac{1}{2\pi} \log(1/r(z, w)) + O(z, w), \quad n = 1.$$

There exists a constant $C$ such that $g(z, w) + C$ is everywhere positive. Define

$$\lambda_z(w) = (g(z, w) + C) \omega_{n-1}(w).$$

Then $dd^c \lambda_z = \omega$. (See Wu [5, I and II].)

Let $a \in A$, and let $(z_1, \ldots, z_n)$ be a holomorphic coordinate system in a neighborhood $U$ centered at $a$, such that in the underlying real coordinate system the metric tensor at $a$ is represented by the identity matrix. Set

$$B(z, w) = |z - w| / r(z, w)$$

where $|z - w|$ is computed in the $(z_1, \ldots, z_n)$ coordinate system. $B$ is continuous and positive on $U \times U$ and $B(a, a) = 1$.

From (1), (2), and (3) we see

$$\lambda_z(w) = |z - w|^{2-2n} O(z, w) \omega_{n-1}, \quad n > 1,$$

(4)

$$\lambda_z(w) = (1/2\pi) \log(1 / |z - w|) + O(z, w), \quad n = 1.$$

From (1), (see Wu[5, II, §8] for justification)
\[ d_w^e \lambda_z(w) = \left[ -\frac{(n-1)!}{2\pi^n} r(z, w)^{2-2n} d_w r(z, w) \right. \]
\[ + O_1(z, w) r(z, w)^{2-2n}) \wedge \omega_{n-1}(w) \quad \text{(if } n > 1) \]
\[ = \left[ \frac{(n-1)!}{2\pi^n} r(z, w)^{2-2n} d_w \log(1/r(z, w)) \right. \]
\[ + O_1(z, w) r(z, w)^{2-2n}) \wedge \omega_{n-1}(w) \quad \text{(any } n). \]

Now \( \log(1/r(z, w)) = \log B(z, w) + \log (1/|z-w|) \) and \( \log B(z, w) \) is locally bounded; hence
\[ d_w^e \lambda_z(w) = \left[ \frac{(n-1)!}{2\pi^n} B(z, w)^{2-2n} r(z, w)^{2-2n} d_w (1/|z-w|) \right. \]
\[ + O_1(z, w) |z-w|^{2-2n}) \wedge \omega_{n-1}(w). \]  

(6)

Since \( d^e \log (1/|z-w|) = -d^e |z-w|/|z-w| \), we have
\[ d^e \lambda_z(w) = O_{2n-1} |z-w|^{1-2n}. \]  

(7)

Finally, look at \( z = 0 \), i.e., at \( a \):
\[ d^e \log(1/|w|) \wedge (\omega_{n-1}(w) - v_{n-1}(w)) = -d^e |w| \wedge (\omega_{n-1}(w) - v_{n-1}(w))/|w|. \]

Since \( \omega_{n-1}(a) = v_{n-1}(a) \), this is bounded. Thus
\[ d^e \lambda_a(w) = \left[ (n-1)!/2\pi^n \right] B(a, w)^{2-2n} |w|^{2-2n} d^e \log(1/|w|) \wedge v_{n-1} \]
\[ + O_{2n-1}(w) |w|^{2-2n}. \]  

(8)

**Lemma 5.** Let \( f: M \to A \) be open and holomorphic, \( K \) a compact subset of \( M \), and \( \chi \) continuous on \( K \); then
\[ F(z) = \int_K f^* \lambda_z \wedge \chi \]
exists for each \( z \in A \) and is continuous.

**Proof.** The question is local on both \( K \) and \( A \), so we may assume \( f(K) \subset U \) and adopt the above notation. Apply (4) and Lemma 1. q.e.d.

**Lemma 6.** In the same situation,
\[ G(z) = \int_K f^* d^e \lambda_z \wedge \chi \]
exists and is continuous on \( A \).

**Proof.** Use (7) and Lemma 1.
3. The Residue Theorem. Let $M$ be a complex manifold of dimension $m = n + q$. Let $A$ be a compact Kähler manifold, and let $f: M \to A$ be an open holomorphic map. Let $H$ be an open subset of $M$. Let $S = \overline{H} - H$ be empty or a boundary manifold of $H$, and suppose that $\overline{H}$ is compact. Let $\psi$ be a continuous function on $\overline{H}$ satisfying a Lipschitz condition. (Then $d\psi$ exists almost everywhere.) Let $\chi$ be a differential form of bidegree $(q, q)$ and class $C^1$ on $M$.

Then $df^*\lambda_a \wedge d\psi \wedge \chi$ and $df^*\lambda_a \wedge \psi \wedge d\chi$ are integrable over $H$ for each $a \in A$.

Also let $T$ be the support of $\psi \chi$ on $S$, and assume that $T \cap f^{-1}(a)$ is a set of measure zero on $f^{-1}(a)$. Suppose that $df^*\lambda_a \wedge \psi \chi$ is integrable on $S$. Then

$$\int_S df^*\lambda_a \wedge \psi \chi = -\int_H df^*\lambda_a \wedge d\psi \wedge \chi - \int_H df^*\lambda_a \wedge \psi \wedge d\chi$$

$$+ \int_D f^*\omega \wedge \psi \chi - \int_{H \cap f^{-1}(a)} v_f \psi \chi.$$

Proof. By Stokes’ Theorem,

$$\int_S \gamma_\rho df^*\lambda_a \wedge \psi \chi = -\int_H \gamma_\rho df^*\lambda_a \wedge d\psi \wedge \chi - \int_H \gamma_\rho df^*\lambda_a \wedge \psi \wedge d\chi$$

$$+ \int_H \gamma_df^*\omega \wedge \psi \chi + \int_H \gamma_\rho \wedge df^*\lambda_a \wedge \psi \chi.$$

We let $\rho \to 0$; from Lemma 6 we see that we need only show that the last integral has limit

$$-\int_{H \cap f^{-1}(a)} v_f \psi \chi.$$

By (8), it is

$$\int_H d\gamma_\rho \wedge [(n - 1)!/2\pi^n] B(a, f)^{2n-2} \left| f \right|^{2-2n} \wedge d^c \log(1/\left| f \right|) \wedge \gamma_n \wedge \psi \chi$$

$$+ \int_H d\gamma_f^* O_{2n-1} \left| f \right|^{2-2n} \wedge \psi \chi.$$

By Lemma 3, the second term has limit zero. By Lemma 4, the first term has limit

$$-\int_{H \cap f^{-1}(a)} v_f \psi \chi B(a, a)^{2n-2}$$

but $B(a, a) = 1$. q.e.d.
Singular Stokes Theorem. Let $M$, $A$, $f$, $H$, $S$, and $\chi$ be as above. Let $\psi$ be a function of class $C^2$ on $\overline{H}$. Then $d^c f^* \lambda_a \wedge d\psi \wedge \chi, f^* \lambda_a \wedge d^d \psi \wedge \chi$, and $f^* \lambda_a \wedge d^c \psi \wedge d\chi$ are integrable over $H$.

Assume further that $f^* \lambda_a \wedge d^c \psi \wedge \chi$ is integrable over $S$. Then

$$\int_S f^* \lambda_a \wedge d^c \psi \wedge \chi = - \int_H d^c f^* \lambda_a \wedge d\psi \wedge \chi + \int_H f^* \lambda_a \wedge d^d \psi \wedge \chi$$

$$- \int_H f^* \lambda_a \wedge d^c \psi \wedge d\chi.$$

Proof. Similar to that of the Residue Theorem. Apply Lemmas 5 and 6; then consider

$$\int_H d\gamma_o \wedge f^* \lambda_a \wedge d^c \psi \wedge \chi.$$

By (4) it is

$$\int_H d\gamma_o \wedge O_o |f|^{2-2n} f^* v_{n-1} \wedge d^c \psi \wedge \chi$$

if $n > 1$ and

$$\int_H d\gamma_o \wedge [(1/2\pi) \log(1/|f|) + O_o] \wedge d^c \psi \wedge \chi$$

if $n = 1$. Now apply Lemma 3. q.e.d.

The so-called Unintegrated Form of the First Main Theorem is obtained by setting $\psi = 1$ and choosing $\chi$ so that $d\chi = 0$ in the Residue Theorem. We can now state the:

Integrated First Main Theorem. Let $M$, $A$, $f$ and $\chi$ be as before, and assume $d\chi = 0$. Let $G$ be an open subset of $M$ with compact closure $\overline{G}$. Assume that $\Gamma = \overline{G} - G$ is a boundary manifold of $G$. Let $g$ be open with $\overline{g} \subseteq G$, and assume that $\gamma = \overline{g} - g$ is a boundary manifold of $g$. Let $\psi$ be a continuous, real-valued function on $M$, of class $C^2$ on $\overline{G} - g$, such that $\psi|_g = R$ a constant, $\psi|_M - G = 0$, and $0 \leq \psi(z) \leq R$ for all $z \in M$.

(Such a $\psi$ satisfies a Lipschitz condition on $M$.) Define

$$m^\delta(a) = - \int_{\Gamma} f^* \lambda_a \wedge d^c \psi \wedge \chi,$$

$$m^\gamma(a) = - \int_{\gamma} f^* \lambda_a \wedge d^c \psi \wedge \chi,$$
\[ N(a) = \int_{f^{-1}(a)} v_r \psi \chi, \]
\[ T = \int_G f^* \omega \wedge \psi \chi, \]
\[ \Delta(a) = \int_G f^* \lambda_a \wedge d^c d\psi \wedge \chi. \]

Then these are continuous functions on \( A \), and
\[ N + m^* - m^b = T + \Delta. \]

Note that \( N, m, m^b \), and \( T \) are always nonnegative; but in general we have information on \( \Delta \).

This theorem is obtained by applying the Residue Theorem to \( G \), the Singular Stokes Theorem to \( G - \tilde{g} \), and eliminating the common term; see [1] for details. The continuity of \( \Delta \) is a consequence of Lemma 5, and the continuity of \( N \) is a theorem of Stoll [3, Theorem 3.8]. Thus \( m^* - m^b \) is continuous. Now let \( h \) be a \( C^\infty \) function with compact support in \( G \) and which is 1 in a neighborhood of \( \tilde{g} \), and replace \( \psi \) by \( h\psi \). We get another FMT with different (but still continuous) \( N, T, \) and \( \Delta \). The new \( m^* \) is zero but the new \( m^b \) is the same as the old. Hence the old \( m^b \) and \( m^* \) are continuous.

**References**


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