

SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

REALIZING HOMEOMORPHISMS BY AMBIENT ISOTOPIES

T. B. RUSHING

An *isotopy* of a topological space X is a continuous map $e: X \times I \rightarrow X$ such that the function $e_t, t \in I$ ($I = [0, 1]$), defined by $e_t(x) = e(x, t)$ is a homeomorphism for all $t \in I$. We call e an *ambient isotopy* if e_0 is the identity. A homeomorphism $h: X \rightarrow X$ is said to be *realized* by an ambient isotopy e if $e_1 = h$. A subspace A of X is *pseudo-isotopic to a point* in X if there is a continuous map $f: A \times I \rightarrow X$ (called a *pseudo-isotopy*) such that if $f_t, t \in I$, denotes the map defined by $f_t(x) = f(x, t)$ then

- (1) f_1 is the inclusion map,
- (2) $f_t, t > 0$, is an embedding of A into X and
- (3) f_0 is a constant map.

The *track* of the pseudo-isotopy f is $f(A \times I)$. An isotopy $e: X \times I \rightarrow X$ is said to be *supported* by a set $Y \subset X$ if $e_t|_{X - Y}$ is the inclusion for all $t \in I$. Finally, the *cone* over a space A is defined to be $(A \times I)/(A \times 0)$.

The following theorem may be considered to be a generalization of [1].

THEOREM. *Let A be a compact subset of a Hausdorff space X where A is pseudo-isotopic to a point in X . Let h be a homeomorphism of X onto itself which is fixed on $X - \text{int } A$. Then, h can be realized by an ambient isotopy which is supported by the track of the pseudo-isotopy.*

PROOF. Let $f: A \times I \rightarrow X$ be the pseudo-isotopy which shrinks A to a point in X . Define $F: A \times I \rightarrow X \times I$ by $F(x, t) = (f(x, t), t)$. Let $\pi: A \times I \rightarrow (A \times I)/(A \times 0)$ be the projection. Then, the function $F\pi^{-1}: (A \times I)/(A \times 0) \rightarrow X \times I$ is one-to-one and continuous. Since $(A \times I)/(A \times 0)$ is compact and $X \times I$ is Hausdorff, $F\pi^{-1}$ is an embedding. Define a (level-preserving) homeomorphism $K: A \times I \rightarrow A \times I$ by $K(a, t) = (\underline{h(a)}, t)$. Then, $\pi K\pi^{-1}$ which carries $(A \times I)/(A \times 0)$ onto

Received by the editors April 4, 1969.

itself is a homeomorphism. Thus, $(F\pi^{-1})_*K\pi^{-1}(F\pi^{-1})^{-1} = FKF^{-1}$ which takes $F(A \times I)$ onto itself is a homeomorphism. Notice that FKF^{-1} is the identity on the frontier of $F(A \times I)$ in $X \times I$.¹ Now define $E: X \times I \rightarrow X \times I$ by $E|_{F(A \times I)} = FKF^{-1}$ and $E|(X \times I) - \text{int } F(A \times I) = 1$. Since $F(A \times I)$ is compact, it is closed in $X \times I$ and so it is easy to see that E is a homeomorphism. Let $e: X \times I \rightarrow X$ be defined by the condition $E(x, t) = (e(x, t), t)$. It is easy to see that e is the desired ambient isotopy.

REFERENCE

1. J. W. Alexander, *On the deformation of an n -cell*, Proc. Nat. Acad. Sci. U.S.A. 9 (1923), 406-407.

THE UNIVERSITY OF GEORGIA

¹ *Added in proof.* Harry Berkowitz pointed out that a condition must be added to the hypothesis of the theorem in order that this always follow. This is easy to do and we leave it to the reader.