ON THE FREE PRODUCT OF RINGS WITH WEAK ALGORITHM

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The following theorem is a special case of [1, Proposition 95], a result which requires highly technical methods. The proof we give is elementary.

Let (R_λ) denote a family of filtered rings, each with weak algorithm, and for each λ let v_λ be a filtration for which R_λ satisfies the weak algorithm. By the criterion [2, p. 335] each R_λ contains a set X_λ and a subdivision ring $K_\lambda = \{a \in R_\lambda : v_\lambda(a) \leqq 0\}$ such that each element has a unique expression

$$(1) \qquad \sum x_I \alpha_I \quad (\alpha_I \in K_\lambda, \text{ a.a. } \alpha_I = 0;\ x_I = x_{i_1} \cdots x_{i_n},\ x_{i_j} \in X_\lambda).$$

The crucial part of the proof is establishing a unique form (1) in the free product.

THEOREM. *Let (R_λ) be a family of filtered rings with weak algorithm as above such that K is the underlying division ring of each R_λ. Then the free product P of (R_λ) over K has a unique filtration extending the filtrations on the factors and P satisfies the weak algorithm for this filtration.*

PROOF. Letting $X = \bigcup X_\lambda$, each element of P may be written in the form (1) with X replacing X_λ. Assume

$$(2) \qquad \sum x_I \alpha_I = \sum x_J \beta_J,$$

write $x_I = x_{I_1} \otimes \cdots \otimes x_{I_t}$ where $x_{I_1} \in R_{\lambda_1}, \cdots, x_{I_t} \in R_{\lambda_t}$ and $\lambda_1 \neq \cdots \neq \lambda_t$. First suppose there is only one summand on the left so (2) may be rewritten as

$$(3) \qquad x_{I_1} \otimes \cdots \otimes x_{I_t} \alpha_I = \sum x_J \beta_J.$$

Now $\sum x_J \beta_J \in x_{I_1} P$, say $\sum x_J \beta_J = x_{I_1} \sum x_{J'} \beta_{J'}$ whence

$$(4) \qquad x_{I_2} \otimes \cdots \otimes x_{I_t} \alpha_I = \sum x_{J'} \beta_{J'}$$

as P is an integral domain. For $t = 1$, $x_{I_1} \in R_{\lambda_1}$ implies the right side of (3) is $x_{I_1} \alpha_I$ and for $t > 1$, (4) provides the induction step so the right side of (3) is $x_{I_1} \otimes \cdots \otimes x_{I_t} \alpha_I$ in each case.

If (2) is arbitrary, by moving some of the summands to the right, (3) is recovered so $x_I \alpha_I = x_J \beta_J$ for some I, J. It follows that the form (1) is unique.

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A valuation may be defined on P as follows. If $x \in X$, $x \in X_\lambda$ for exactly one λ and we let $v(x) = v_\lambda(x)$. For $x_I = x_{i_1} \cdots x_{i_n}$, $x_{i_j} \in X$, define $v(x_I) = \sum_1^n v(x_{i_j})$ and finally extend v to all of P by defining $v(\sum x_I \alpha_I) = \max\{v(x_I) : \alpha_I \neq 0\}$, $v(\alpha) = 0$ if $\alpha \in K^*$ and $v(0) = -\infty$. It is immediate that v is a filtration of P which uniquely extends each v_λ. The remaining conditions that P satisfy the weak algorithm for v may be easily established by the reader.

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REFERENCES

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