A PRODUCT INTEGRAL REPRESENTATION FOR A GRONWALL INEQUALITY

BURRELL W. HELTON

1. Introduction. This paper shows that if $G$ and $H$ are functions from $R \times R$ to $R$ such that $1 - G \geq c > 0$, $G$ and $H$ are integrable and have bounded variation on $[a, b]$, $f$ is bounded and $f(x) \leq k + (LR)\int_a^b (fH + fG)$ for $x \in [a, b]$ then

(1) if $G \geq 0$ and $H \geq 0$, then $f(x) \leq k \prod_a^x (1 + H)(1 - G)^{-1}$ and $k \prod_a^x (1 + H)(1 - G)^{-1}$ is a solution of the inequality,

(2) if $1 - |G| \geq c > 0$ and $\sup \neq 0$, then $f(x) \leq k \prod_a^x (1 + |H|)(1 - |G|)^{-1}$, and

(3) if $k \geq 0$, the requirement $1 - G \geq c > 0$ cannot be relaxed. Also a Gronwall-type inequality is stated and proved for functions $f$, $G$ and $H$ which have ranges in a normed ring.

The Main Theorem of Schmaedeke and Sell [3] is a special case of Theorem 4 of this paper. The linear function $J(f)$ defined by Herod [2] is more general than the function $J(f) = (LR)\int (fH + fG)$ defined above; however, there are linear functions $(LR)\int (fH + fG)$ which will satisfy the hypothesis of Theorem 4 but will not satisfy the hypothesis of Herod’s theorem.

2. Definitions and preliminary theorems. For detailed definitions, see [1, p. 299]. All sum and product integrals (represented by the symbol $\prod_a^x G$) are subdivision-refinement-type limits of appropriate sums or products: $(LR)\int_a^b (fH + fG) \sim f(x)H(x, y) + f(y)G(x, y)$, $(m)\int_a^b G \sim \frac{1}{2} [f(x) + f(y)]G(x, y)$, $\prod_a^x (1 + H)(1 - G)^{-1} \sim [1 + H(x, y)][1 - G(x, y)]$, etc. and it is understood that $a \leq x < y \leq b$; $R$ is the set of real numbers, and $N$ is a ring which has a multiplicative element 1 and has a norm $| \cdot |$ with respect to which $N$ is complete and $|1| = 1$; $f, u, v, G, H$ are functions from $R$ or $R \times R$ to $N$. $G \in OA^0$ on $[a, b]$ iff $\int_a^b G$ exists and $\int_a^b G - fG = 0$; $G \in OM^0$ on $[a, b]$ iff $\prod_a^x (1 + G)$ exists for $a \leq x < y \leq b$ and $\int_a^b (1 + G) - \prod (1 + G) = 0$; $G \in OL^0$ iff $\lim_{x \to p^-} G(x, p)$, $\lim_{x \to p^+} G(p, x)$, $\lim_{x,y \to p^-} G(x, y)$ and $\lim_{x,y \to p^+} G(p, x)$ exist for $p \in [a, b]$. The function $G$ is bounded on $[a, b]$ means there is a subdivision $\{x_i\}_0^n$ of $[a, b]$ and a number $M$ such that if $0 < i \leq n$ and $x_{i-1} \leq x < y \leq x_i$ then $|G(x, y)| < M$. A similar meaning is given to each statement such as $G > 0$ on $[a, b]$, $(1 - G)^{-1}$ exists on $[a, b]$, etc.

Received by the editors May 5, 1969.

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Theorem 1. Given: $f$ and $h$ are functions from $R$ to $N$ and $H, G$ and $B$ are functions from $R \times R$ to $N$ such that on $[a, b]$, $h$ has bounded variation, $(1 - G)^{-1}$ exists and is bounded, $dh(1 - G)^{-1} \in OA^0$, $B = (1 + H)(1 - G)^{-1}$, $B - 1$ has bounded variation and $\prod B$ exists for $a \leq x < y \leq b$.

Conclusion. The following two statements are equivalent:

1. $f(x)H(x, y) + f(y)G(x, y) \in OA^0$ and $f(x) = h(x) + (LR)f_a(fH + fG)$ for $x \in [a, b]$; and

2. if $a \leq x < y \leq b$, then $(LR)f_a[f(t)B - \prod B] = 0$ and $f(y) = f(x)\prod B + (R)f_a dh(1 - G)^{-1}\prod B$.

This theorem is a special case of Theorem 5.1 [1, p. 310].

Theorem 2. If $H$ and $G$ are functions from $R \times R$ to $N$ such that $H \in OL^0$, $G \in OA^0$ and $G$ has bounded variation on $[a, b]$, then $GH$ and $HG \in OA^0$ and $OM^0$ on $[a, b]$. Furthermore, if $H$ has bounded variation and $H \in OA^0$ on $[a, b]$ then $\int_a^y GH = \sum_{x \in S} G(x^-, x)H(x^-, x) + G(x, x^+)(x^+, x)$, where $S$ is the subset of $[a, b]$ such that $x \in S$ iff $G$ has a discontinuity at $x$, $G(a^-, a) = 0$ and $G(b, b^+) = 0$.

The proof of this theorem is given in §4.

If $f, H, G$ are functions such that on $[a, b]$, $f, H, G$ have bounded variation, $H \in OA^0, G \in OA^0$, and $(1 - G)^{-1}$ exists and is bounded, then $(1 - G)^{-1} \in OL^0$ on $[a, b]$ and it follows from Theorem 2 that $df(1 - G)^{-1}$, $H(1 - G)^{-1}$, $fH$, $(1 + H)(1 - G)^{-1} - 1 = (H + G)(1 - G)^{-1} \in OA^0$ and $OM^0$ on $[a, b]$.

3. The principal results.

Theorem 3. $H$ and $G$ are functions of bounded variation from $R \times R$ to $R$, $c \in R$, $H \in OA^0$, $G \in OA^0$, $H \geq 0$, $G \geq 0$ and $1 - G \geq c > 0$ on $[a, b]$ and $u$ is a function from $R$ to $R$ such that $u$ is bounded above on $[a, b]$, $(LR)f_a^*(uH + uG)$ exists and $u(x) \leq f_a^*(uH + uG)$ for $x \in [a, b]$.

Conclusion. If $x \in [a, b]$, then $u(x) \leq 0$.

Proof. Assume the conclusion is false and let $S$ be the subset of $[a, b]$ such that $x \in S$ iff $u(x) > 0$; then $S$ is nonempty and has a greatest lower bound $p$. Since

$$u(p) \leq (LR) \int_a^p (uH + uG) + (LR) \int_p^p (uH + uG) \leq (LR) \int_p^p (uH + uG) \leq u(p)G(p^-, p),$$

then $u(p)[1 - G(p^-, p)] \leq 0$ and $u(p) \leq 0$; furthermore, $p < b$.

Since $H$ and $G$ have bounded variation and since $G(p, p^+) < 1$, then
there is a number \( y, p < y \leq b \), such that \( \int_a^y H + \int_a^y G < \frac{1}{2} [G(p, p^+)] + 1 \).
Let \( M \) be the least upper bound for \( u \) on \([p, y]\); then there is a number \( z \in [p, y] \) such that \( u(z) > \frac{1}{2} M [G(p, p^+) + 1] \). Hence,

\[
\begin{align*}
u(z) & \leq (LR) \int_a^z (uH + uG) = (LR) \left( \int_a^p + \int_p^{p^+} + \int_{p^+}^x \right) (uH + uG) \\
& \leq u(p) H(p, p^+) + MG(p, p^+) + \int_{p^+}^z (MH + MG) \\
& \leq M \left( \int_{p^+}^z H + \int_p^z G \right) < \frac{1}{2} M [G(p, p^+) + 1] < u(z).
\end{align*}
\]
This contradiction proves that \( u(x) \leq 0 \) for \( x \in [a, b] \).

**Theorem 4.** Given: \( H \) and \( G \) are functions of bounded variation from \( R \times R \) to \( R \), \( c \in R \), \( H \in OA^\circ \), \( G \in OA^\circ \) and \( 1 - G \geq c > 0 \) on \([a, b]\) and \( f \) is a function from \( R \) to \( R \) such that \( f \) is bounded above on \([a, b]\), \((LR)\int_a^b (fH + fG) \) exists, \( k \) is a number and \( f(x) \leq k + (LR) \int_a^x (fH + fG) \) for \( x \in [a, b] \).

**Conclusion.** (1) If \( H \geq 0 \) and \( G \geq 0 \) on \([a, b]\), then

\[
f(x) \leq k \prod_a^x (1 + H)(1 - G)^{-1} = k \prod_a^x (1 + H) / \prod_a^x (1 - G)
\]
for \( a \leq x \leq b \). Furthermore, the function \( f(x) = k \prod_a^x (1 + H)(1 - G)^{-1} \) is a solution to the inequality.

(2) If \( c \in R \), \( 1 - |G| \geq c > 0 \), \((LR)\int_a^b (|H| + |G|) \) exists and \( f \geq 0 \) on \([a, b]\), then

\[
f(x) \leq k \prod_a^x (1 + |H|)(1 - |G|)^{-1} = k \prod_a^x (1 + |H|) / \prod_a^x (1 - |G|)
\]
for \( a \leq x \leq b \).

**Proof of part 1.** Suppose \( H \geq 0 \) and \( G \geq 0 \) on \([a, b]\). Since \((1 - G)^{-1} \) exists and is bounded on \([a, b]\), it follows from Theorem 2 that \((1+H)(1-G)^{-1} - 1 \in OM^\circ \) and has bounded variation on \([a, b]\). Let \( v \) be the function such that \( v(x) = k \prod_a^x (1+H)(1-G)^{-1} \) for \( x \in [a, b] \); then \( v \) is bounded on \([a, b]\). It follows from Theorem 1(2\(\rightarrow 1\)) that \( v(x) = k + (LR) \int_a^x (vH + vG) \). Let \( u = f - v \); then, for \( x \in [a, b] \),

\[
u(x) \leq (LR) \int_a^x [(f - v)H + (f - v)G] = (LR) \int_a^x (uH + uG).
\]
Since $f$ is bounded above, then $u$ is bounded above on $[a, b]$ and, from Theorem 3, $u(x) \leq k \prod \epsilon^2 (1 + H)(1 - G)^{-1}
abla = k \prod \epsilon^2 (1 + H)/(1 - G)$, since $\prod \epsilon^2 (1 - G) \neq 0$. The second half of part 1 follows because $k \prod \epsilon^2 (1 + H)(1 - G)^{-1}$ is a solution of the equation $f(x) = k + (LR) \int_a^b (fH + fG)$.

**Proof of Part 2.** Suppose $1 - |G| \geq c > 0$, $(LR) \int_a^b (f|H| + f|G|)$ exists and $f \geq 0$ on $[a, b]$. Since $|H| \in O\mathcal{A}$ and $|G| \in O\mathcal{A}$ and

$$f(x) \leq k + (LR) \int_a^b (fH + fG) \leq k + (LR) \int_a^b (f|H| + f|G|)$$

for $x \in [a, b]$, the desired inequality follows from part 1 above. Note that if $f$ is quasicontinuous, it follows from Theorem 2 that $(LR) \int_a^b (f|H| + f|G|)$ exists.

**Theorem 5.** If $H$ and $G$ are functions from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, $G(b^-, b) \geq 1$, and $k$ and $M$ are nonnegative numbers, then there is a function $f$ such that $f(x) \leq k + (LR) \int_a^b (fH + fG)$ for $x \in [a, b]$ and $f(b) > kM$.

**Proof.** Let $f$ be the function such that $f = 0$ on $[a, b)$ and $f(b) > kM$. Then

$$f(b) = f(b)G(b^-, b) = (LR) \int_a^b (fH + fG) \leq k + (LR) \int_a^b (fH + fG),$$

and, if $x \in [a, b)$, $f(x) = 0 \leq k + (LR) \int_a^b (fH + fG)$.

**Theorem 6.** If $M, k$ and $c$ are numbers such that $M > 0$ and $c > 0$ and $H$ and $G$ are functions from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ such that $G(b^-, b) > 1$ and on $[a, b]$ $H$ and $G$ have bounded variation, $H \in O\mathcal{A}$, $G \in O\mathcal{A}$ and $|1 - G| > c$, then there is a function $f$ from $\mathbb{R}$ to $\mathbb{R}$ such that $f(b) > |k| M$ and $f(x) \leq k + (LR) \int_a^b (fH + fG)$ for $x \in [a, b]$.

**Proof.** Let $f$ be a function such that $f(x) = k \prod \epsilon^2 (1 + H)(1 - G)^{-1}$ for $x \in [a, b)$ and $f(b)$ is a number such that $f(b) > |k| M$ and $f(b)[G(b^-, b) - 1] + [(LR) \int_a^b fH + (R) \int_a^b fG + k] > 0$. From Theorem 4, $f$ is a solution on $[a, b)$. Also,

$$(LR) \int_a^b (fH + fG) = (L) \int_a^b fH + (R) \int_a^b fG + f(b)[G(b^-, b) - 1]$$

+ $f(b) + k - k > f(b) - k$;

therefore, $f$ is a solution on $[a, b]$. In the following theorem, $A$ and $B$ denote the functions $B = (1 + H)(1 - G)^{-1}$ and $A(p, q) = \prod \epsilon^2 B$; $P$ denotes a bound for $A$ on $[a, b]$; $Q(x, y) = G(x, y)[1 - G(x, y)]^{-1}P$ and $M(a, x)$ is the sum of the
magnitudes of the discontinuities of $Q$ on $[a, b]$. Note that $N$ is a normed ring and that the inequalities $|G(x^-, x)| > 1$ and $|G(x, x^+)| > 1$ are permitted.

**Theorem 7.** Given: $k > 0$, $f$ is a function from $R$ to $N$, $G$ and $H$ are functions from $R \times R$ to $N$ such that on $[a, b]$ $f$, $G$ and $H$ have bounded variation, $G \in OA^0$, $H \in OA^0$, and $(1 - G)^{-1}$ exists and is bounded.

Conclusion. If $|f(x) - (LR)f_a^a(fH + fG)| < k$ for $a \leq x \leq b$, then

$$|f(y)| \leq k[1 + V_a^a A + 2M(a, y)]$$

for $a \leq y \leq b$.

Proof. Let $h$ be the function such that $h(y) = f(y) - (LR)f_a^a(fH + fG)$ for $a \leq y \leq b$. Since $f$, $H$ and $G$ have bounded variation, then $h$ has bounded variation. The function $B - 1 = (1 + H)(1 - G)^{-1} - 1 = (H + G)(1 - G)^{-1}$ has bounded variation. From Theorem 2, $dH(1 - G)^{-1} \in OA^0$, $f(x)H(x, y) + f(y)G(x, y) \in OA^0$, $B - 1 \in OM^0$, and $\prod B$ exists for $a \leq x < y \leq b$. Since $f(y) = h(y) + (LR)f_a^a(fH + fG)$, the hypothesis of Theorem 1 $(1 \rightarrow 2)$ is satisfied and for $a \leq y \leq b$

$$f(y) = f(a)A(a, y) + (R)\int_a^y dh(1 - G)^{-1}A(t, y)$$

$$= f(a)A(a, y) + (R)\int_a^y dh[1 + G(1 - G)^{-1}]A(t, y)$$

$$= f(a)A(a, y) + (R)\int_a^y dhA(t, y) + (R)\int_a^y dhG(1 - G)^{-1}A(t, y),$$

and

$$f(a)A(a, y) + (R)\int_a^y dhA(t, y)$$

$$= f(a)A(a, y) + h(t)A(t, y)\bigg|_a^y - (L)\int_a^y hdA(t, y)$$

$$= f(a)A(a, y) + h(y)A(y, y) - h(a)A(a, y) - (L)\int_a^y hdA(t, y)$$

$$= h(y) - (L)\int_a^y hdA(t, y).$$

From Theorem 2, it follows that $| (R)\int_a^y dh(G(1 - G)^{-1}A(t, y)) | \leq 2kM(a, y)$. Hence,
\[ |f(y)| \leq \left| h(y) - (L) \int_a^y hA(t, y) + (R) \int_a^y dh[G(1 - G)^{-1}A(t, y)] \right| \]
\[ \leq k + kV_aA + 2kM(a, y) = k[1 + V_aA + 2M(a, y)]. \]

If \( H(x, y) = G(x, y) = \frac{1}{2}[g(y) - g(x)] \), then \( (m)Jf \) is a special case of \( (LR)fH + fG \), \( (m)Jf \) is a special case of \( (LR)fH + fG \), and Schmaedeke and Sell's Main Theorem [3, p. 1219] is a special case of Theorem 4. Similarly, \( (R)Jfdg \), \( (L)Jfdg \) and the Riemann-Stieltjes integral are special cases. If \( f \) is left or right continuous on \([a, b]\), then \( (D)Jfdg = (R)Jfdg \) or \( (D)Jfdg = (L)Jfdg \), respectively, where \( (D)Jfdg \) is the Dushkin integral [3, p. 1218]. Herod's linear function \( J(f) [2, p. 570] \) is more general than the function \( J(f)(x, y) = (LR)fH + fG \); however, the results of Theorem 4 are better than Herod's results in the sense that Theorem 4 permits \( f \) to have unbounded variation and permits \( (LR)f^H + fG \) to have discontinuities greater than 1. Note that the function \( \prod (1 + H)(1 - G)^{-1} \) defined in Theorem 4 satisfies each of the properties listed by Herod for the function \( m(x, y) \): \( m(x, y) \geq 1 \), \( m(x, y)m(y, z) = m(x, z) \) for \( x < y < z \), and \( m(0, x) = 1 + J[m(0, \cdot)](0, x) \).

4. **Proof of Theorem 2.** In this section Theorem 2 and a necessary lemma are proved.

**Lemma.** Given: \( H \) is a function from \( RXR \) to \( N \), \( H \in OL^o \) on \([a, b]\), \( e > 0 \), and \( S^- \) and \( S^+ \) are subsets of \([a, b]\) such that \( p \in S^- \) iff \( \lim_{x,y \to p^-} H(x, y) - H(p^-, p) | \geq e \)

and \( p \in S^+ \) iff \( \lim_{x,y \to p^+} H(x, y) - H(p, p^+) | \geq e \).

**Conclusion.** (1) \( S^- \) and \( S^+ \) are finite sets and (2) there is a subdivision \( \{x_i\}_0^n \) of \([a, b]\) such that \( H \) is bounded on \([x_i-1, x_i]\) for \( i = 1, 2, 3, \cdots, n \).

**Proof.** Suppose \( S^- \) is an infinite set; then \( S^- \) has an accumulation point \( q \in [a, b] \) and there is a subset \( \{p_n\}_1^\infty \) of \( S^- \) and a sequence \( \{x_n, y_n\}_1^\infty \) of number pairs such that \( p_n \to q^- \) and \( x_n, y_n \to q^- \) (or \( p_n \to q^+ \) and \( x_n, y_n \to q^+ \)) and such that \( |H(x_n, y_n) - H(p_n, p_n)| \geq e \) for \( n = 1, 2, 3, \cdots \). Since \( H \in OL^o \),

\[ \lim_{n \to \infty} H(x_n, y_n) = \lim_{n \to \infty} H(p_n, p_n) \]

and...
\[ 0 = \lim_{n \to \infty} \left| H(x_n, y_n) - H(p_n^-, p_n^+) \right| \geq \epsilon. \]

Similarly, \( S^+ \) is a finite set.

Since \( H \in \mathcal{O}L^0 \), then \( H \) is bounded in a neighborhood of each point of \([a, b]\). The covering theorem assures that there is a subdivision which has the desired property.

**Proof of Theorem 2.** Let \( \epsilon > 0 \) and let \( M \) be the number and \( A, B, C, D, E, T_i \) be the number sets defined as follows:

1. \( A = \{a_i\}_{i=0}^n \) is a subdivision of \([a, b]\) and \( M \) is a number such that if \( 0 < i \leq r \) and \( a_{i-1} \leq x < y \leq a_i \) then \( |H(x, y)| < M \).
2. \( B = \{b_i\}_{i=1}^s \) is the subset of \([a, b]\) such that \( p \in B \) iff
   \[
   \lim_{x, y \to p^+} H(x, y) - H(p, p^+) \geq \epsilon/(8V_{aG})
   \]
   or
   \[
   \lim_{x, y \to p^-} H(x, y) - H(p^-, p) \leq \epsilon/(8V_{bG}).
   \]
3. \( C = \{c_i\}_{i=1}^s \) and \( D = \{d_i\}_{i=1}^s \) are subsets of \([a, b]\) such that \( c_i < b_i < d_i \) for \( i = 1, 2, \ldots, s \) and
   \[
   \sum_{i=1}^s (V_{c_i}^{b_i}G + V_{d_i}^{b_i}G) < \epsilon/8M
   \]
   and \( |H(x, b_i) - H(y, b_i)|, |H(b_i, x) - H(b_i, y)| \) and \( |H(x, y) - H(p, q)| \) are less than \( \epsilon/(8V_{aG}) \) whenever \( x, y, p, q \in [c_i, b_i] \) or \( x, y, p, q \in (b_i, d_i] \).
4. \( T_i = \{t_{ij}\}_{j=1}^r \) for \( i = 1, 2, \ldots, s \) is a subdivision of \([d_i, c_{i+1}]\) such that, if \( t_{ij} \leq x < y \leq t_{i,j+1} \) and \( t_{ij} \leq p < q \leq t_{i,j+1} \), then \( |H(x, y) - H(p, q)| < \epsilon/(8V_{bG}) \).
5. \( E = \{z_i\}_{i=1}^m \) is a subdivision of \([a, b]\) such that if \( D' = \{y_{ij}\}_{i,j} \) is a refinement of \( E \), then
   \[
   \sum_i \left| G_i - \sum_j G_{ij} \right| < \epsilon/8M,
   \]
   where \( G_i = G(z_{i-1}, z_i) \) and \( G_{ij} = G(y_{i,j-1}, y_{ij}) \) and \( z_{i-1} \leq y_{i,j-1} < y_{ij} \leq z_i \).

Similar abbreviated notations are used in the following manipulations.

Let \( D' = \{x_{ij}\}_{i,j} \) be a refinement of the subdivision \( K = A \cup B \cup C \cup D \cup E \cup T_i = \{x_i\}_i \). In the following, \( \sum_i \) depends on \( K \); \( \sum_i G_{ij} \) depends on \( D' \) and \( \{x_{i-1}, x_i\}; i \in Q \) iff \( x_i \in B \).
\[ \sum_i \left| \sum_j H_{ij}G_{ij} - H_iG_i \right| \]

\[ = \sum_i \left| H_i \left( \sum_j G_{ij} - G_i \right) + \sum_j (H_{ij} - H_i)G_{ij} \right| \]

\[ \leq \sum_i \left| H_i \right| \cdot \left| \sum_j G_{ij} - G_i \right| + \sum_{i \in Q} \sum_j \left| (H_{ij} - H_i)G_{ij} \right| \]

\[ + \sum_{i \in Q} \sum_j \left| (H_{ij} - H_i) \right| \left| G_{ij} \right| \]

\[ < M(\epsilon/8M) + \sum_{i \in Q} \sum_j b_{ij} \left| H_{ij} - H_i \right| \left| G_{ij} \right| \]

\[ + \sum_{i \in Q} \sum a_{ij} \left| H_{ij} - H_i \right| \left| G_{ij} \right| + (\epsilon/8V_{aG})V_{bG}^b \]

\[ < \epsilon/8 + (\epsilon/8V_{aG})V_{bG}^b + 2M \sum_i (V_{cG}^b G + V_{dG}^b G) + \epsilon/8 < \epsilon, \]

where \( a_{ij} = 1 \) and \( b_{ij} = 0 \) provided \( i \in Q \) and \( x_{ij} \) is the largest element of \( D' \) such that \( x_{ij} < x_i \) or \( x_{ij} \) is the smallest element of \( D' \) such that \( x_{ij} > x_i \); otherwise, \( a_{ij} = 0 \) and \( b_{ij} = 1 \). Hence, \( HG \in OA_0 \) and, similarly, \( GH \in OA_0 \). It follows from Theorem 3.4 [1, p. 301] that \( HG \) and \( GH \in OM_0 \).

Suppose \( H \) has bounded variation and \( H \in OA_0 \) on \([a, b]\) and let \( g \) and \( h \) be the functions such that \( g(x) = G(a, x) \) and \( h(x) = H(a, x) \); then \( g \) and \( h \) are quasicontinuous and it follows from Theorem 3.1 [1, p. 300] that

\[ \int_a^b GH = \int_a^b (fG)(fH) = \int_a^b dgdh \]

\[ = \sum_{x \in S} \{ [g(x) - g(x^-)][h(x) - h(x^-)] \]

\[ + [g(x^+) - g(x)][h(x^+) - h(x)] \} \].

**Bibliography**


Southwest Texas State University