A PRODUCT INTEGRAL REPRESENTATION FOR A GRONWALL INEQUALITY

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1. **Introduction.** This paper shows that if G and H are functions from $R\times R$ to R such that $1-G\geqq c>0$, G and H are integrable and have bounded variation on $[a, b]$, f is bounded and $f(x)\leqq k+(LR)\int_a^x(fH+fG)$ for $x\in[a, b]$ then

(1) if $G\geqq 0$ and $H\geqq 0$, then $f(x)\leqq k\prod_a^x(1+H)(1-G)^{-1}$ and $k\prod_a^x(1+H)(1-G)^{-1}$ is a solution of the inequality,

(2) if $1-|G|\geqq c>0$ and $f\geqq 0$, then $f(x)\leqq k\prod_a^x(1+|H|)(1-|G|)^{-1}$, and

(3) if $k\geqq 0$, the requirement $1-G\geqq c>0$ cannot be relaxed. Also a Gronwall-type inequality is stated and proved for functions f, G and H which have ranges in a normed ring.

The Main Theorem of Schmaedeke and Sell [**3**] is a special case of Theorem 4 of this paper. The linear function $J(f)$ defined by Herod [**2**] is more general than the function $J(f)=(LR)\int(fH+fG)$ defined above; however, there are linear functions $(LR)\int_x^y(fH+fG)$ which will satisfy the hypothesis of Theorem 4 but will not satisfy the hypothesis of Herod's theorem.

2. **Definitions and preliminary theorems.** For detailed definitions, see [**1**, p. 299]. All sum and product integrals (represented by the symbol $\prod_a^b G$) are subdivision-refinement-type limits of appropriate sums or products: $(LR)\int_a^b(fH+fG)\sim f(x)H(x, y)+f(y)G(x, y)$, $(m)\int_a^b Gf\sim\frac{1}{2}[f(x)+f(y)]G(x, y)$, $\prod_a^b(1+H)(1-G)^{-1}\sim[1+H(x, y)][1-G(x, y)]^{-1}$, etc. and it is understood that $a\leqq x<y\leqq b$; R is the set of real numbers, and N is a ring which has a multiplicative element 1 and has a norm $|\cdot|$ with respect to which N is complete and $|1|=1$; f, u, v, G, H are functions from R or $R\times R$ to N. $G\in OA^\circ$ on $[a, b]$ iff $\int_a^b G$ exists and $\int_a^b|G-\int G|=0$; $G\in OM^\circ$ on $[a, b]$ iff $\prod_x^y(1+G)$ exists for $a\leqq x<y\leqq b$ and $\int_a^b|(1+G)-\prod(1+G)|=0$; $G\in OL^\circ$ iff $\lim_{x\to p^-}G(x, p)$, $\lim_{x\to p^+}G(p, x)$, $\lim_{x,y\to p^-}G(x, y)$ and $\lim_{x,y\to p^+}G(p, x)$ exist for $p\in[a, b]$. The function G is bounded on $[a, b]$ means there is a subdivision $\{x_i\}_0^n$ of $[a, b]$ and a number M such that if $0<i\leqq n$ and $x_{i-1}\leqq x<y\leqq x_i$ then $|G(x, y)|<M$. A similar meaning is given to each statement such as $G>0$ on $[a, b]$, $(1-G)^{-1}$ exists on $[a, b]$, etc.

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THEOREM 1. *Given: f and h are functions from R to N and H, G and B are functions from $R\times R$ to N such that on $[a, b]$, h has bounded variation, $(1-G)^{-1}$ exists and is bounded, $dh(1-G)^{-1}\in OA^\circ$, $B=(1+H)(1-G)^{-1}$, $B-1$ has bounded variation and $\prod_x^y B$ exists for $a\leqq x<y\leqq b$.*

CONCLUSION. The following two statements are equivalent:

(1) $f(x)H(x,y)+f(y)G(x,y)\in OA^\circ$ and $f(x)=h(x)+(LR)\int_a^x(fH+fG)$ for $x\in[a, b]$; and

(2) if $a\leqq x<y\leqq b$, then $(L)\int_a^y|f(t)[B-\prod B]|=0$ and $f(y)=f(x)\prod_x^y B+(R)\int_x^y dh(1-G)^{-1}\prod_t^y B$.

This theorem is a special case of Theorem 5.1 [1, p. 310].

THEOREM 2. *If H and G are functions from $R\times R$ to N such that $H\in OL^\circ$, $G\in OA^\circ$ and G has bounded variation on $[a, b]$, then GH and $HG\in OA^\circ$ and OM° on $[a, b]$. Furthermore, if H has bounded variation and $H\in OA^\circ$ on $[a, b]$ then $\int_a^b GH=\sum_{x\in S} G(x^-, x)H(x^-, x)+G(x, x^+)H(x, x^+)$, where S is the subset of $[a, b]$ such that $x\in S$ iff G has a discontinuity at x, $G(a^-, a)=0$ and $G(b, b^+)=0$.*

The proof of this theorem is given in §4.

If f, H, G are functions such that on $[a, b]$, f, H, G have bounded variation, $H\in OA^\circ$, $G\in OA^\circ$, and $(1-G)^{-1}$ exists and is bounded, then $(1-G)^{-1}\in OL^\circ$ on $[a, b]$ and it follows from Theorem 2 that $df(1-G)^{-1}$, $H(1-G)^{-1}$, fH, $(1+H)(1-G)^{-1}-1=(H+G)(1-G)^{-1}\in OA^\circ$ and OM° on $[a, b]$.

3. The principal results.

THEOREM 3. *H and G are functions of bounded variation from $R\times R$ to R, $c\in R$, $H\in OA^\circ$, $G\in OA^\circ$, $H\geqq 0$, $G\geqq 0$ and $1-G\geqq c>0$ on $[a, b]$ and u is a function from R to R such that u is bounded above on $[a, b]$, $(LR)\int_a^b(uH+uG)$ exists and $u(x)\leqq\int_a^x(uH+uG)$ for $x\in[a, b]$.*

CONCLUSION. If $x\in[a, b]$, then $u(x)\leqq 0$.

PROOF. Assume the conclusion is false and let S be the subset of $[a, b]$ such that $x\in S$ iff $u(x)>0$; then S is nonempty and has a greatest lower bound p. Since

$$u(p)\leqq(LR)\int_a^{p^-}(uH+uG)+(LR)\int_{p^-}^{p}(uH+uG)$$

$$\leqq(LR)\int_{p^-}^{p}(uH+uG)\leqq u(p)G(p^-, p),$$

then $u(p)[1-G(p^-, p)]\leqq 0$ and $u(p)\leqq 0$; furthermore, $p<b$.

Since H and G have bounded variation and since $G(p, p^+)<1$, then there is a number y, $p<y\leqq b$, such that $\int_{p^+}^{y}H+\int_{p}^{y}G<\frac{1}{2}[G(p,\ p^+)+1]$. Let M be the least upper bound for u on $[p, y]$; then there is a number $z\in[p,\ y]$ such that $u(z)>\frac{1}{2}M[G(p,\ p^+)+1]$. Hence,

$$u(z) \leqq (LR)\int_a^z (uH + uG) = (LR)\left(\int_a^p + \int_p^{p^+} + \int_{p^+}^z\right)(uH + uG)$$

$$\leqq u(p)H(p,\ p^+) + MG(p,\ p^+) + \int_{p^+}^z (MH + MG)$$

$$\leqq M\left(\int_{p^+}^z H + \int_p^z G\right) < \tfrac{1}{2}M[G(p,\ p^+) + 1] < u(z).$$

This contradiction proves that $u(x)\leqq 0$ for $x\in[a,\ b]$.

THEOREM 4. *Given*: *H and G are functions of bounded variation from $R\times R$ to R, $c\in R$, $H\in OA^\circ$, $G\in OA^\circ$ and $1-G\geqq c>0$ on $[a,\ b]$ and f is a function from R to R such that f is bounded above on $[a,\ b]$, $(LR)\int_a^b (fH+fG)$ exists, k is a number and $f(x)\leqq k+(LR)\int_a^x(fH+fG)$ for $x\in[a, b]$.*

CONCLUSION. (1) If $H\geqq 0$ and $G\geqq 0$ on $[a,\ b]$, then

$$f(x) \leqq k\prod_a^x (1 + H)(1 - G)^{-1} = k\prod_a^x (1 + H) \Big/ \prod_a^x (1 - G)$$

for $a\leqq x\leqq b$. Furthermore, the function $f(x)=k\prod_a^{x-} (1+H)(1-G)^{-1}$ is a solution to the inequality.

(2) If $c\in R$, $1-|G|\geqq c>0$, $(LR)\int_a^b(f|H|+f|G|)$ exists and $f\geqq 0$ on $[a,\ b]$, then

$$f(x) \leqq k\prod_a^x (1 + |H|)(1 - |G|)^{-1}$$

$$= k\prod_a^x (1 + |H|) \Big/ \prod_a^x (1 - |G|)$$

for $a\leqq x\leqq b$.

PROOF OF PART 1. Suppose $H\geqq 0$ and $G\geqq 0$ on $[a,\ b]$. Since $(1-G)^{-1}$ exists and is bounded on $[a,\ b]$, it follows from Theorem 2 that $(1+H)(1-G)^{-1}-1\in OM^\circ$ and has bounded variation on $[a,\ b]$. Let v be the function such that $v(x)=k\prod_a^x (1+H)(1-G)^{-1}$ for $x\in[a, b]$; then v is bounded on $[a,\ b]$. It follows from Theorem 1(2→1) that $v(x)=k+(LR)\int_a^x (vH+vG)$. Let $u=f-v$; then, for $x\in[a,\ b]$,

$$u(x) \leqq (LR)\int_a^x [(f - v)H + (f - v)G] = (LR)\int_a^x (uH + uG).$$

Since f is bounded above, then u is bounded above on $[a, b]$ and, from Theorem 3, $u(x) \leqq 0$; hence, $f(x) \leqq v(x) = k\prod_a^x (1+H)(1-G)^{-1} = k\prod_a^x(1+H)/\prod_a^x(1-G)$, since $\prod_a^x(1-G) \neq 0$. The second half of part 1 follows because $k\prod_a^x(1+H)(1-G)^{-1}$ is a solution of the equation $f(x) = k + (LR)\int_a^x(fH+fG)$.

PROOF OF PART 2. Suppose $1-|G| \geqq c > 0$, $(LR)\int_a^b(f|H|+f|G|)$ exists and $f \geqq 0$ on $[a, b]$. Since $|H| \in OA^\circ$ and $|G| \in OA^\circ$ and

$$f(x) \leqq k + (LR)\int_a^x (fH + fG) \leqq k + (LR)\int_a^x (f|H| + f|G|)$$

for $x \in [a, b]$, the desired inequality follows from part 1 above. Note that if f is quasicontinuous, it follows from Theorem 2 that $(LR)\int_a^b (f|H|+f|G|)$ exists.

THEOREM 5. *If H and G are functions from $R \times R$ to R, $G(b^-, b) \geqq 1$, and M and k are nonnegative numbers, then there is a function f such that $f(x) \leqq k + (LR)\int_a^x(fH+fG)$ for $x \in [a, b]$ and $f(b) > kM$.*

PROOF. Let f be the function such that $f = 0$ on $[a, b)$ and $f(b) > kM$. Then

$$f(b) \leqq f(b)G(b^-, b) = (LR)\int_a^b (fH + fG) \leqq k + (LR)\int_a^b (fH + fG),$$

and, if $x \in [a, b)$, $f(x) = 0 \leqq k + (LR)\int_a^x(fH+fG)$.

THEOREM 6. *If M, k and c are numbers such that $M > 0$ and $c > 0$ and H and G are functions from $R \times R$ to R such that $G(b^-, b) > 1$ and on $[a, b]$ H and G have bounded variation, $H \in OA^\circ$, $G \in OA^\circ$ and $|1-G| > c$, then there is a function f from R to R such that $f(b) > |k|M$ and $f(x) \leqq k + (LR)\int_a^x(fH+fG)$ for $x \in [a, b]$.*

PROOF. Let f be a function such that $f(x) = k\prod_a^x(1+H)(1-G)^{-1}$ for $x \in [a, b)$ and $f(b)$ is a number such that $f(b) > |k|M$ and $f(b)[G(b^-, b) - 1] + [(L)\int_a^b fH + (R)\int_a^{b^-} fG + k] > 0$. From Theorem 4, f is a solution on $[a, b)$. Also,

$$(LR)\int_a^b (fH + fG) = (L)\int_a^b fH + (R)\int_a^{b^-} fG + f(b)[G(b^-, b) - 1]$$
$$+ f(b) + k - k > f(b) - k;$$

therefore, f is a solution on $[a, b]$.

In the following theorem, A and B denote the functions $B = (1+H)(1-G)^{-1}$ and $A(p, q) = \prod_p^q B$; P denotes a bound for A on $[a, b]$; $Q(x, y) = G(x, y)[1-G(x, y)]^{-1}P$ and $M(a, x)$ is the sum of the magnitudes of the discontinuities of Q on $[a, b]$. Note that N is a normed ring and that the inequalities $|G(x^-, x)| > 1$ and $|G(x, x^+)| > 1$ are permitted.

THEOREM 7. *Given*: $k > 0$, *f is a function from R to N, G and H are functions from $R \times R$ to N such that on $[a, b]$ f, G and H have bounded variation, $G \in OA^\circ$, $H \in OA^\circ$, and $(1-G)^{-1}$ exists and is bounded.*

CONCLUSION. If $|f(x) - (LR)\int_a^x (fH+fG)| < k$ for $a \leqq x \leqq b$, then

$$|f(y)| \leqq k[1 + V_a^y A + 2M(a, y)]$$

for $a \leqq y \leqq b$.

PROOF. Let h be the function such that $h(y) = f(y) - (LR)\int_a^y (fH+fG)$ for $a \leqq y \leqq b$. Since f, H and G have bounded variation, then h has has bounded variation. The function $B - 1 = (1+H)(1-G)^{-1} - 1 = (H+G)(1-G)^{-1}$ has bounded variation and, from Theorem 2, $dh(1-G)^{-1} \in OA^\circ$, $f(x)H(x, y) + f(y)G(x, y) \in OA^\circ$, $B - 1 \in OM^\circ$, and $\prod_x^y B$ exists for $a \leqq x < y \leqq b$. Since $f(y) = h(y) + (LR)\int_a^y (fH+fG)$, the hypothesis of Theorem 1(1→2) is satisfied and for $a \leqq y \leqq b$

$$\begin{aligned} f(y) &= f(a)A(a, y) + (R)\int_a^y dh(1 - G)^{-1}A(t, y) \\ &= f(a)A(a, y) + (R)\int_a^y dh[1 + G(1 - G)^{-1}]A(t, y) \\ &= f(a)A(a, y) + (R)\int_a^y dhA(t, y) + (R)\int_a^y dhG(1 - G)^{-1}A(t, y), \end{aligned}$$

and

$$\begin{aligned} &f(a)A(a, y) + (R)\int_a^y dhA(t, y) \\ &\qquad = f(a)A(a, y) + h(t)A(t, y)\Big|_a^y - (L)\int_a^y hdA(t, y) \\ &\qquad = f(a)A(a, y) + h(y)A(y, y) - h(a)A(a, y) - (L)\int_a^y hdA(t, y) \\ &\qquad = h(y) - (L)\int_a^y hdA(t, y). \end{aligned}$$

From Theorem 2, it follows that $|(R)\int_a^y dh[G(1-G)^{-1}A(t, y)]| \leqq 2kM(a, y)$. Hence,

$$|f(y)| \leqq \left| h(y) - (L)\int_a^y hdA(t, y) + (R)\int_a^y dh[G(1-G)^{-1}A(t, y)] \right|$$

$$\leqq k + kV_a^y A + 2kM(a, y) = k[1 + V_a^y A + 2M(a, y)].$$

If $H(x, y) = G(x, y) = \frac{1}{2}[g(y) - g(x)]$, then $(m)\int fdg = (LR)\int (fH + fG)$, $(m)\int fdg$ is a special case of $(LR)\int (fH+fG)$, and Schmaedeke and Sell's Main Theorem [3, p. 1219] is a special case of Theorem 4. Similarly, $(R)\int fdg$, $(L)\int fdg$ and the Riemann-Stieltjes integral are special cases. If f is left or right continuous on $[a, b]$, then $(D)\int fdg = (R)\int fh$ or $(D)\int fdg = (L)\int fG$, respectively, where $(D)\int fdg$ is the Dushkin integral [3, p. 1218]. Herod's linear function $J(f)$ [2, p. 570] is more general than the function $J(f)(x, y) = (LR)\int_x^y (fH + fG)$; however, the results of Theorem 4 are better than Herod's results in the sense that Theorem 4 permits f to have unbounded variation and permits $(LR)\int_x^y (fH+fG)$ to have discontinuities greater than 1. Note that the function $\prod_x^y (1+H)(1-G)^{-1}$ defined in Theorem 4 satisfies each of the properties listed by Herod for the function $m(x, y)$: $m(x, y) \geqq 1$, $m(x, y)m(y, z) = m(x, z)$ for $x < y < z$, and $m(0, x) = 1 + J[m(0, \cdot)](0, x)$.

4. **Proof of Theorem** 2. In this section Theorem 2 and a necessary lemma are proved.

LEMMA. *Given: H is a function from $R \times R$ to N, $H \in OL^\circ$ on $[a, b]$, $e > 0$, and S^- and S^+ are subsets of $[a, b]$ such that $p \in S^-$ iff*

$$\left| \lim_{x,y \to p^-} H(x, y) - H(p^-, p) \right| \geqq e$$

and

$$p \in S^+ \qquad \text{iff} \left| \lim_{x,y \to p^+} H(x, y) - H(p, p^+) \right| \geqq e.$$

CONCLUSION. (1) S^- and S^+ are finite sets and (2) there is a subdivision $\{x_i\}_0^n$ of $[a, b]$ such that H is bounded on $[x_{i-1}, x_i]$ for $i = 1, 2, 3, \cdots, n$.

PROOF. Suppose S^- is an infinite set; then S^- has an accumulation point $q \in [a, b]$ and there is a subset $\{p_n\}_1^\infty$ of S^- and a sequence $\{x_n, y_n\}_1^\infty$ of number pairs such that $p_n \to q^-$ and $x_n, y_n \to q^-$ (or $p_n \to q^+$ and $x_n, y_n \to q^+$) and such that $|H(x_n, y_n) - H(p_n^-, p_n)| \geqq e$ for $n = 1, 2, 3, \cdots$. Since $H \in OL^\circ$,

$$\lim_{n\to\infty} H(x_n, y_n) = \lim_{n\to\infty} H(p_n^-, p_n)$$

and

$$0 = \lim_{n\to\infty} \mid [H(x_n, y_n) - H(p_n^-, p_n)] \mid \geqq e.$$

Similarly, S^+ is a finite set.

Since $H \in OL^\circ$, then H is bounded in a neighborhood of each point of $[a, b]$. The covering theorem assures that there is a subdivision which has the desired property.

PROOF OF THEOREM 2. Let $\epsilon > 0$ and let M be the number and A, B, C, D, E, T_i be the number sets defined as follows:

(1) $A = \{a_i\}_0^r$ is a subdivision of $[a, b]$ and M is a number such that if $0 < i \leqq r$ and $a_{i-1} \leqq x < y \leqq a_i$ then $|H(x, y)| < M$.

(2) $B = \{b_i\}_1^s$ is the subset of $[a, b]$ such that $p \in B$ iff

$$\left| \lim_{x,y\to p^+} H(x, y) - H(p, p^+) \right| \geqq \epsilon/(8V_a^bG)$$

or

$$\left| \lim_{x,y\to p^-} H(x, y) - H(p^-, p) \right| \geqq \epsilon/(8V_a^bG).$$

(3) $C = \{c_i\}_1^s$ and $D = \{d_i\}_1^s$ are subsets of $[a, b]$ such that $c_i < b_i < d_i$ for $i = 1, 2, \cdots, s$ and

$$\sum_1^s (V_{c_i}^{b_i^-} G + V_{b_i^+}^{d_i} G) < \epsilon/8M$$

and $|H(x, b_i) - H(y, b_i)|$, $|H(b_i, x) - H(b_i, y)|$ and $|H(x, y) - H(p, q)|$ are less than $\epsilon/(8V_a^bG)$ whenever $x, y, p, q \in [c_i, b_i)$ or $x, y, p, q \in (b_i, d_i]$.

(4) $T_i = \{t_{ij}\}_j$ for $i = 1, 2, \cdots, s$ is a subdivision of $[d_i, c_{i+1}]$ such that, if $t_{ij} \leqq x < y \leqq t_{i,j+1}$ and $t_{ij} \leqq p < q \leqq t_{i,j+1}$, then $|H(x, y) - H(p, q)| < \epsilon/(8V_a^bG)$.

(5) $E = \{z_i\}_1^m$ is a subdivision of $[a, b]$ such that if $D' = \{y_{ij}\}_{i,j}$ is a refinement of E, then

$$\textstyle\sum_i |G_i - \sum_j G_{ij}| < \epsilon/8M,$$

where $G_i = G(z_{i-1}, z_i)$ and $G_{ij} = G(y_{i,j-1}, y_{ij})$ and $z_{i-1} \leqq y_{i,j-1} < y_{ij} \leqq z_i$. Similar abbreviated notations are used in the following manipulations.

Let $D' = \{x_{ij}\}_{i,j}$ be a refinement of the subdivision $K = A \cup B \cup C \cup D \cup E \cup_1^s T_i = \{x_i\}_i$. In the following, $\sum_i$ depends on K; $\sum_j G_{ij}$ depends on D' and $[x_{i-1}, x_i]$; $i \in Q$ iff $x_i \in B$.

$$
\begin{aligned}
\sum_i \Big| \sum_j H_{ij}G_{ij} - H_iG_i \Big| & \\
&= \sum_i \Big| H_i\Big(\sum_j G_{ij} - G_i \Big) + \sum_j (H_{ij} - H_i)G_{ij} \Big| \\
&\leqq \sum_i | H_i | \cdot \Big| \sum_j G_{ij} - G_i \Big| + \sum_{i\in Q} \sum_j | (H_{ij} - H_i)G_{ij} | \\
&\quad + \sum_{i\notin Q} \sum_j | (H_{ij} - H_i) | | G_{ij} | \\
&< M(\epsilon/8M) + \sum_{i\in Q} \sum_j b_{ij} | H_{ij} - H_i | | G_{ij} | \\
&\quad + \sum_{i\in Q} \sum_j a_{ij} | H_{ij} - H_i | | G_{ij} | + (\epsilon/8V_a^bG)V_a^bG \\
&< \epsilon/8 + (\epsilon/8V_a^bG)V_a^bG + 2M \sum_i (V_{c_i}^{b_i^-} G + V_{b_i^+}^{d_i}G) + \epsilon/8 < \epsilon,
\end{aligned}
$$

where $a_{ij}=1$ and $b_{ij}=0$ provided $i\in Q$ and x_{ij} is the largest element of D' such that $x_{ij}<x_i$ or x_{ij} is the smallest element of D' such that $x_{ij}>x_i$; otherwise, $a_{ij}=0$ and $b_{ij}=1$. Hence, $HG\in OA^\circ$ and, similarly, $GH\in OA^\circ$. It follows from Theorem 3.4 [**1**, p. 301] that HG and $GH\in OM^\circ$.

Suppose H has bounded variation and $H\in OA^\circ$ on $[a, b]$ and let g and h be the functions such that $g(x)=G(a, x)$ and $h(x)=H(a, x)$; then g and h are quasicontinuous and it follows from Theorem 3.1 [**1**, p. 300] that

$$
\begin{aligned}
\int_a^b GH = \int_a^b (\textstyle\int G)(\int H) = \int_a^b dgdh & \\
= \sum_{x\in S} \{ [g(x) - g(x^-)][h(x) - h(x^-)] & \\
& + [g(x^+) - g(x)][h(x^+) - h(x)] \}.
\end{aligned}
$$

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