THE KERNEL OF A STARSHAPED SUBSET OF THE PLANE

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A point p of a subset V of Euclidean two-dimensional space $\boldsymbol{R}^2$ is a *star-center* for V provided the line segment $[p, q] \equiv \{tp+(1-t)q \mid 0 \leqq t \leqq 1\} \subset V$ for each $q \in V$. The set V is *starshaped* if it possesses at least one star-center and the set H of all star-centers of V is called the *kernel* of V. It is an elementary fact that H must be convex. Also it is easy to show that if V is closed then H is closed. We will say that a point $\gamma(t_0)$ on a curve $\gamma: (a, b) \to \boldsymbol{R}^2$, $a < t_0 < b$, is an inflection point provided there is no $\delta > 0$ such that $\gamma(t_0-\delta, t_0+\delta) - \{\gamma(t_0)\}$ is completely contained in one of the two open half-planes determined by the tangent line of γ at $\gamma(t_0)$.

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THEOREM 1. *If V is a closed starshaped subset of the plane such that the boundary ∂V of V is a continuously differentiable simple closed curve, then ∂H is contained in the union of ∂V and the tangent lines to inflection points on ∂V. If it is also assumed that ∂V is a closed curve with a finite number M of inflection points and $\partial H \cap \partial V = \emptyset$, then ∂H is a polygon with at most M sides.*

PROOF. First we will show that closure (int V) $= V$ where int V denotes the interior of V. (The referee has pointed out that if we assume that V is compact, then the Jordan Schoenflies Theorem implies closure (int V) $= V$.) Since ∂V is a simple closed curve $X - \partial V$ consists of two open connected components, one bounded, inside ∂V, and one unbounded, outside ∂V. An easy connectedness argument shows that if $V \cap$ inside $\partial V \neq \emptyset$ then inside $\partial V \subset V$ and similarly if $V \cap$ outside $\partial V \neq \emptyset$ then outside $\partial V \subset V$. Since V is closed, $\partial V \subset V$ and hence the only possibilities for V are: ∂V, $\partial V \cup$ inside ∂V, $\partial V \cup$ outside ∂V, $\boldsymbol{R}^2$. $V = \boldsymbol{R}^2$ is eliminated because then $\partial V = \emptyset$. ∂V and $\partial V \cup$ outside ∂V are eliminated because each is not star-shaped. To see this just prolong a line from a proposed starcenter through a point of inside ∂V until it hits ∂V as it must since inside ∂V is bounded. Thus $V = \partial V \cup$ inside ∂V and hence closure (int V) $= V$ as we wished to show.

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Suppose $q\in\partial H-\partial V$. We will show that $q\in T_p$ for some $p\in\partial V$ where T_p denotes the tangent line of ∂V drawn through p. Since $q\in\partial H$ there is a sequence $q_n\in R^2-H$ such that $q_n\to q$. Because $q\in\partial H-\partial V$ we have $q\in\text{int } V$, and so $q_n\in\text{int } V$ for n sufficiently large. Consequently we may assume $q_n\in\text{int } V$ for all n. Next we will show that for each n there is a point $r_n\in\text{int } V$ such that $[q_n, r_n]\not\subset V$. To see this assume the contrary. Then int $V\subset\{r\mid [q_n, r]\subset V\}$ and since the latter set is clearly closed we must have $V=\text{closure (int } V)$ $\subset\{r\mid [q_n, r]\subset V\}$. But this contradicts $q_n\notin H$ and so such an r_n must exist. Clearly, $r_n\neq q_n$. Now note that the point q cannot be on $l(r_n, q_n)$ $\equiv$the line through r_n and q_n because $q\in l(r_n, q_n)$ would imply $[r_n, q_n]\cup[q, r_n]\cup[q, q_n]\subset V$, a contradiction. Thus r_n, q_n and q are not colinear. Now we will swing a line segment about the point q_n until it hits ∂V. To do this analytically we let t be the greatest number such that $[q_n, sr_n+(1-s)q]\subset V$ for all s, $0\leqq s\leqq t$. Since V is closed, t must exist. Next we will show that $t>0$. From $q_n\in\text{int } V$ and $q\in H$ it follows easily that $[q_n, q)\equiv\{(1-t)q_n+tq\mid 0\leqq t<1\}\subset\text{int } V$. Similarly $[r_n, q)\subset\text{int } V$. Also noting that $q\in\text{int } V$ we can conclude $[q_n, q]\cup[q, r_n]\subset\text{int } V$. It follows that $t>0$. Now set $s_n=tr_n+(1-t)q$. Then q_n, s_n and q are not colinear. Consider the triangle $\Delta=\text{convex hull}$ $\{q_n, q, s_n\}$. Clearly $\Delta\subset V$ and int $\Delta\neq\emptyset$. It is also clear that there must be a point $p_n\in\partial V\cap(q_n, s_n)$ where

$$(q_n, s_n) \equiv \{tq_n + (1 - t)s_n \mid 0 < t < 1\}.$$

Next $l(q_n, s_n)$ must be tangent to ∂V at p_n for otherwise a portion of ∂V would extend into int $\Delta\subset V$ and would drag along with it some points of R^2-V. Thus $q_n\in T_{p_n}$. Since ∂V is a closed curve it is compact and thus a subsequence of $\{p_n\}$, which we may take to be $\{p_n\}$ itself, will converge to a point $p\in\partial V$. We will show that the line determined by p and q is tangent to ∂V at p. Denote the angle made by a line l with the x-axis by Al. Now because ∂V is continuously differentiable we have $Al(p, q)=\lim_n Al(p_n, q_n)=\lim_n AT_{p_n}=AT_p$ where we have used the fact that $q\neq p(q\in\text{int } V, p\in\partial V)$ in the first equality. It follows that $l(p, q)=T_p$ and thus $q\in T_p$ as we wished to show.

Next we will show that p is an inflection point of ∂V. We may assume without loss of generality that the origin is at p and the positive x-axis equals the ray from p through q. An open neighborhood N of p in ∂V can clearly be parameterized by a map $\gamma\colon(-\epsilon, \epsilon)\to R^2$ such that

$$\gamma(t) = (\gamma_x(t), \gamma_y(t)) = (t, \gamma_y(t))$$

with $\gamma_v(0)=0$ for some $\epsilon>0$. Clearly

$$\left.\frac{d\gamma_v}{dt}\right|_{t=0} = 0$$

and so ϵ may be chosen small enough to insure

$$\left|\frac{d\gamma_v}{dt}\right|_t < 1 \qquad \text{for } -\epsilon < t < \epsilon.$$

Since $p\notin\partial V-N$ and $\partial V-N$ is compact we can find a δ, $0<\delta<\epsilon$, such that $Q=\{(x, y)\,|\,|x|<\delta, |y|<\delta\}$ does not intersect $\partial V-N$. Clearly, from the way ∂V is given in N we can say that $Q\cap\partial V =\{(x,\gamma_v(x))\,|\,|x|<\delta\}$. The sets $A=\{(x,y)\,|\,|x|<\delta, |y|<\delta, y>\gamma_v(x)\}$ and $B=\{(x, y)\,|\,|x|<\delta, |y|<\delta, y<\gamma_v(x)\}$ are easily seen to be connected and they obviously do not intersect ∂V. Consequently each is entirely in either V or $\boldsymbol{R}^2-V$. Since $p=(0, 0)\in\partial V$ we must have either $A\subset V$ and $B\subset\boldsymbol{R}^2-V$ or $A\subset\boldsymbol{R}^2-V$ and $B\subset V$. By reversing the y-axis if necessary we may assume $A\subset V$ and $B\subset\boldsymbol{R}^2-V$. Now assume p is not an inflection point of ∂V. Then by definition we have a δ_1, $0<\delta_1<\delta$ such that either $\gamma_v(x)>0$ for $0<|x|<\delta_1$, or $\gamma_v(x)<0$ for $0<|x|<\delta_1$. Since the x-axis from $p=(0, 0)$ to q is in V and $B\subset\boldsymbol{R}^2-V$ we must have the second possibility. Pick an x such that $-\delta_1<x<0$. Since $q\in H$ we must have $[(x, \gamma_v(x)), q]\subset V$. But the intersection of $[(x, \gamma_v(x)), q]$ and the negative y-axis is obviously a point of B. This is a contradiction and so we must admit that p is an inflection point of ∂V. This establishes the first conclusion of the theorem.

In the case where the number M of inflection points of ∂V is finite and $\partial H\cap\partial V=\varnothing$ we know that ∂H is the boundary of the convex set H and is contained in M lines. It is not difficult to show that this implies that ∂H is a convex polygon. We leave the details to the reader. Q.E.D.

Continue to assume that V is a starshaped subset of the plane with ∂V a continuously differentiable closed curve. Locally the curve ∂V has an interior side and exterior side and so does each tangent line. Thus for each $p\in\partial V$ we may define H_p to be the close half-plane determined by T_p and on the interior side of T_p.

THEOREM 2. *If* int $H\neq\varnothing$ *then* $H=\tilde{H}\equiv\bigcap H_p\cap V$, *$p$ an inflection point.*

PROOF. First suppose p is an inflection point of ∂V and $q\notin H_p$. Then as in the second to last paragraph of the proof of Theorem 1, we may justify the picture of the situation given in Figure 1, and the conclusion that $[p, q] \not\subset V$. It follows that $H \subset H_p$ and $H \subset \tilde{H}$.

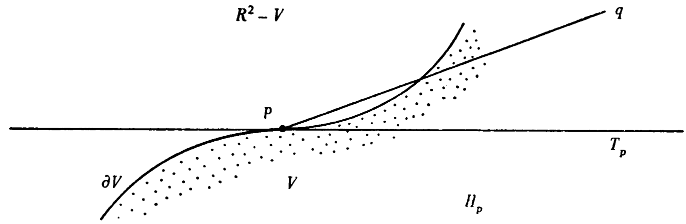

FIGURE 1

Now let $q_0 \in \operatorname{int} H \subset \operatorname{int} \tilde{H}$. Let q be an arbitrary point of $\tilde{H}$. Suppose $q \notin H$.

Then since $q_0 \in \operatorname{int} H$ and $q \in V$ we can conclude that $[q_0, q) \subset \operatorname{int} V$. Since $q \notin H$ and H is closed, there must be a point $r \in (q_0, q) \cap \partial H$. It follows from $[q_0, q) \subset \operatorname{int} V$ and Theorem 1 that $r \in T_p$ for some inflection point p of ∂V. If $T_p \neq l(q_0, q)$ then either $q_0 \notin H_p$ or $q \notin H_p$. Both possibilities are absurd. But if $T_p = l(q_0, q)$ then q_0 would be the limit of a sequence of points in $\boldsymbol{R}^2 - H_p \subset \boldsymbol{R}^2 - \tilde{H} \subset \boldsymbol{R}^2 - H$ which would contradict $q_0 \in \operatorname{int} H$. We have reached a contradiction in all

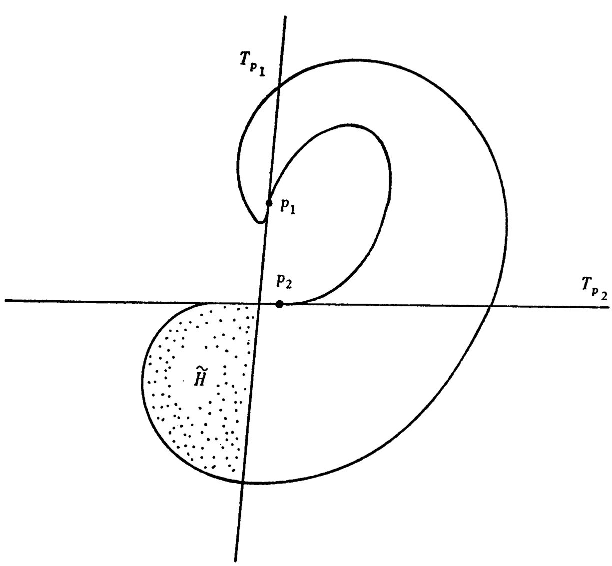

FIGURE 2

cases and hence $q \notin H$ is untenable. Therefore $\tilde{H} \subset H$ and hence $H = \tilde{H}$ as we wished to prove. Q.E.D.

It is not true that $H = \tilde{H}$ without any assumptions. For example, Figure 2 shows a set V (not starshaped) for which $H = \emptyset$ and $\tilde{H} \neq \emptyset$. Hence $\tilde{H}$ cannot be used to tell whether V is starshaped or not. But in order to obtain the funny behavior in this example it *seems* necessary that V not be starshaped. This leads to the following conjecture.

CONJECTURE. If $V \subset \mathbf{R}^2$ is starshaped and ∂V is a continuously differentiable closed curve then $H = \tilde{H}$.

OPEN QUESTION. Must $\tilde{H}$ always be convex?

BIBLIOGRAPHY

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