EQUIVALENT METRICS GIVING DIFFERENT VALUES TO METRIC-DEPENDENT DIMENSION FUNCTIONS1

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In [1] K. Nagami and J. H. Roberts introduced metric-dependent dimension functions \( d_2 \) and \( d_3 \) defined on the class of all metric spaces. All definitions are given below. The definition of \( d_5 \) is due to R. E. Hodel [5]. The following relations hold for all metric spaces \((X, \rho)\):
\[
d_2(X, \rho) \leq d_3(X, \rho) \leq d_5(X, \rho) \leq \mu\dim(X, \rho) \leq \dim X,
\]
where \( \mu\dim \) is metric dimension as defined by Katětov [3], and \( \dim X \) is covering dimension.

The following is a natural question. Suppose \( d \) is \( d_2, d_3, d_5, \) or \( \mu\dim \); and suppose \( d(X, \rho) = r < n = \dim X \). Then for every \( k (r \leq k \leq n) \) does there exist a topologically equivalent metric \( \rho_k \) for \( X \) such that \( d(X, \rho_k) = k \)? Roberts and Slaughter [2] answered this question in the affirmative when \( d \) is \( \mu\dim \). Roberts [6] answered this question in the affirmative for all separable metric spaces when \( d \) is \( d_3 \). This paper answers this question in the affirmative for all metric spaces when \( d \) is \( d_3 \) or \( d_5 \). The question remains unanswered when \( d \) is \( d_2 \).

In the following if \( S \) is a set, \( |S| \) will denote the cardinality of \( S \).

**Definition.** Let \( \eta \) be any ordinal number. A metric space \((X, \rho)\) is said to have property \( P(|\eta|, n, \rho) \) if given any collection of pairs of closed sets indexed by \( \eta \), \( \mathcal{C} = \{(C_\alpha, C'_\alpha): \alpha < \eta\} \) such that there exists an \( \epsilon > 0 \) with \( \rho(C_\alpha, C'_\alpha) \leq \epsilon \) for all \( \alpha < \eta \) then there exists a collection of closed sets \( \{B_\alpha: \alpha < \eta\} \) such that \( B_\alpha \) separates \( X \) between \( C_\alpha \) and \( C'_\alpha \) and order \( \{B_\alpha: \alpha < \eta\} \leq k \).

**Definition.** \( d_2(X, \rho) \) is the smallest integer \( n \) such that \((X, \rho)\) has property \( P(n+1, n, \rho) \).

**Definition.** \( d_3(X, \rho) \) is the smallest integer \( n \) such that \((X, \rho)\) has property \( P(m, n, \rho) \) for every integer \( m \).

**Definition.** \( d_5(X, \rho) \) is the smallest integer \( n \) such that \((X, \rho)\) has property \( P(\aleph_0, n, \rho) \).

**Definition.** \( \mu\dim(X, \rho) \) is the smallest integer \( n \) such that for each \( \epsilon > 0 \) there exists an open cover \( \mathcal{U} \) of \( X \) with

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EQUIVALENT METRICS GIVING DIFFERENT VALUES

(i) \( p \)-mesh \( \mathcal{U} \leq \epsilon \) and
(ii) order \( \mathcal{U} \leq n + 1 \).

**Lemma.** Suppose \((X, \rho)\) is a metric space, \(f: X \to [0, 1]\) is a continuous function and \(\sigma(x, y) = \rho(x, y) + |f(x) - f(y)|\). Then \(\sigma\) is a metric on \(X\) topologically equivalent to \(\rho\). (See [4, p. 199].)

**Theorem 1.** Suppose \((X, \rho)\) is a metric space, \(f: X \to [0, 1]\) a continuous function and \(\sigma(x, y) = \rho(x, y) + |f(x) - f(y)|\).

1. If \(\eta\) is any ordinal number such that \(\aleph_0 \leq |\eta| \leq 2^{\aleph_0}\) and \((X, \rho)\) has property \(P(|\eta|, k, \rho)\) then \((X, \sigma)\) has property \(P(|\eta|, k + 1, \sigma)\).
2. If \((X, \rho)\) has property \(P(m, k, \rho)\) for every integer \(m\) then \((X, \sigma)\) has property \(P(m, k + 1, \sigma)\) for every integer \(m\).

**Remark.** To prove Theorem 2, the principle result of this paper, Theorem 1 is used only for the cases where \(\eta\) is countable or finite.

**Proof of Theorem 1.** Let \(\mathcal{E} = \{(C_a, C'_a): \alpha < \eta\}\) be any collection of pairs of closed sets with \(|\eta| \geq 2^{\aleph_0}\) and with \(\sigma(C_a, C'_a) \geq \epsilon\) for all \(\alpha < \eta\) and for some \(\epsilon > 0\). Choose an integer \(N_0\) so that \(1/N_0 < \epsilon/4\). Since \(|\eta| \leq 2^{\aleph_0}\) there exists a set \(T_0\) with \(T_0 \subseteq [0, 1/N_0]\) and with \(|T_0| = |\eta|\). We can assume that \(T_0 = \{t_\alpha: \alpha < \eta\}\) where if \(t_\alpha = t_\beta\) then \(\alpha = \beta\). Let \(S = \{0, 1, 2, \ldots, N_0\}\) and let \(t_0 = 0\) and \(t_{N_0 + 1} = 1\) for all \(\alpha < \eta\). For each \(i, 1 \leq i \leq N_0\) and each \(\alpha < \eta\) let \(t_i^\alpha = t_\alpha + (i - 1)/N_0\). Now for each \(i \in S\) and each \(\alpha < \eta\) define

\[
E_i^\alpha = \{x: t_i^\alpha \leq f(x) \leq t_{i+1}^\alpha\}
\]

and define

\[
D_i^\alpha = \{x: \rho((C_a \cap E_i^\alpha), x) \geq \epsilon/4\}.
\]

Then for \(i \in S\) and \(\alpha < \eta\) we have \(\rho((C_a \cap E_i^\alpha), C_a \cap E_i^\alpha) \geq 3\epsilon/4\) because for \(x, y \in E_i^\alpha\) we have \(|f(x) - f(y)| \leq 1/N_0 \leq \epsilon/4\). Thus \(D_i^\alpha \supseteq C_a \cap E_i^\alpha\) for all \(i \in S\) and \(\alpha < \eta\). Now \(\mathcal{D} = \{(D_i^\alpha, C_a \cap E_i^\alpha): i \in S, \alpha < \eta\}\) is a collection of pairs of closed sets with \(\rho(D_i^\alpha, C_a \cap E_i^\alpha) \geq \epsilon/4\). In Case (1) where \(\aleph_0 \leq |\eta| \leq 2^{\aleph_0}\), we have \(|\mathcal{D}| = |\eta|\). In Case (2) we have \(|\eta| = m\) for some integer \(m\) and \(|\mathcal{D}| = m\aleph_0\). Thus in either case our hypothesis guarantees the existence of a collection of closed sets \(\{B_i^\alpha: i \in S, \alpha < \eta\}\) with

(i) order \(\{B_i^\alpha: i \in S, \alpha < \eta\} \leq k\) and
(ii) \(X - B_i^\alpha = U_i^\alpha \cup V_i^\alpha\) where \(U_i^\alpha\) and \(V_i^\alpha\) are disjoint open sets and \(D_i^\alpha \subseteq V_i^\alpha\) and \(C_a \cap E_i^\alpha \subseteq U_i^\alpha\).

For each \(\alpha < \eta\) we will modify the collection \(\{B_i^\alpha: i \in S\}\) to obtain a closed set \(B_\alpha\) separating \(X\) between \(C_a\) and \(C'_a\) using a variation of a method due to J. H. Roberts [7].

For each \(i \in S\) and \(\alpha < \eta\) define \(L_i^\alpha = \{x: f(x) = t_i^\alpha\}\). Notice that
\[ L^i = E^i_{a-1} \cap E^i_a, \text{ for } i \text{ such that } 1 \leq i \leq N_0. \] 
For each \( \alpha < \eta \), let \( L^N_0 + 1 = \emptyset \) and define
\[
B_a = \bigcup_{j=0}^{N_0} \left[ \left( B_a^j \cap E_a^j \right) \cup \left( L_a^{j+1} \cap \left( \left( U_a^j - U_a^{j+1} \right) \cup \left( U_a^{j+1} - U_a^j \right) \right) \right) \right],
\]
\[
U_a = \bigcup_{j=0}^{N_0} \left( U_a^j \cap E_a^j \right) - B_a,
\]
\[
V_a = \bigcup_{j=0}^{N_0} \left( V_a^j \cap E_a^j \right) - B_a.
\]

**Assertion 1.** For each \( \alpha < \eta \), \( B_a \) is a closed set separating \( X \) between \( C_a \) and \( C'_a \).

**Proof.** First we show that \( B_a \) is closed. Let \( H_a = \bigcup_{j=0}^{N_0} \left( B_a^j \cap E_a^j \right) \) and let \( G_a = \bigcup_{j=0}^{N_0} \left[ \left( U_a^j \cap E_a^j \right) \cup \left( L_a^{j+1} \cap \left( \left( U_a^j - U_a^{j+1} \right) \cup \left( U_a^{j+1} - U_a^j \right) \right) \right) \right] \). It suffices to show that \( G_a \subseteq B_a \) since \( H_a \) is closed. If \( x \) is a limit point of \( G_a \) then there exists some \( k \in S \) such that \( x \) is a limit point of \( L_a^{k+1} \cap \left( \left( U_a^k - U_a^{k+1} \right) \cup \left( U_a^{k+1} - U_a^k \right) \right) \). We may assume then that \( x \) is a limit point of \( U_a^k \) hence a limit point of \( U_a^{k-1} \). But \( X - B_a^k = U_a^k \cup V_a^k \). Thus either \( x \in U_a^k \) or \( x \in B_a^k \) and in either case \( x \in B_a \) so \( B_a \) is closed.

Next we show that \( X - B_a = U_a \cup V_a \). If \( x \in X \), there exists \( k \in S \) such that \( x \in E_a^k \), since \( \bigcup_{j=0}^{N_0} E_a^j = X \). If \( x \in B_a \) then surely \( x \in B_a^k \cap E_a^k \). But \( X - B_a^k = U_a^k \cup V_a^k \) so \( x \) is in one of \( U_a^k \) or \( V_a^k \) hence one of \( U_a \) or \( V_a \).

We show that \( U_a \cap V_a = \emptyset \). If \( x \in U_a \) then either \( x \in U_a^k \) for exactly one \( k \in S \) or \( x \in E_a^k \cap E_a^{k+1} \) for exactly one \( k \in S \). In the first case since \( x \in U_a^k \), we have \( x \in V_a^k \) hence \( x \in V_a \). In the second case we can show that \( x \in U_a^k \). The only possibility to have \( x \in V_a \) is to have \( x \in V_a^{k+1} \). But then \( x \in U_a^{k+1} \) hence \( x \in \left( U_a^k - U_a^{k+1} \right) \cap \left( E_a^k \cap E_a^{k+1} \right) \). Thus \( x \in B_a \) and \( x \in V_a \) so we conclude that \( U_a \cap V_a = \emptyset \).

To show that \( C_a \subseteq U_a \) we first show that \( C_a \cap B_a = \emptyset \). Let \( x \in C_a \) and suppose that \( x \in E_a^j \) for exactly one \( k \). Then \( x \in L_a^j \) for any \( j \in S \).

Now \( (C_a \cap E_a^j) \cap B_a^k = \emptyset \) because \( X - B_a^k = U_a^k \cup V_a^k \) where \( (C_a \cap E_a^j) \subseteq U_a^k \). So in this case \( x \in B_a \). If \( x \in E_a^k \cap E_a^{k+1} \) for some \( k \in S \) then \( x \in U_a^k \) and \( x \in U_a^{k+1} \). Thus \( x \in B_a \) and \( C_a \cap B_a = \emptyset \). Since \( C_a \subseteq \bigcup_{j=0}^{N_0} \left( U_a^j \cap E_a^j \right) \) and \( C_a \cap B_a = \emptyset \) we conclude that \( C_a \subseteq U_a \).

From the definition of \( D_a^j \) it is clear that \( C'_a \subseteq \bigcup_{j=0}^{N_0} D_a^j \) but \( D_a^j \subseteq V_a \) so \( C'_a \subseteq \bigcup_{j=0}^{N_0} V_a \). Thus if we show that \( C'_a \cap B_a = \emptyset \) we can conclude that \( C'_a \subseteq V_a \). Let \( x \in C'_a \) and let \( x \in E_a^k \) for exactly one \( k \). Then \( x \in V_a^k \) hence \( x \in B_a \). If \( x \) is in \( E_a^k \cap E_a^{k+1} \) then \( x \in V_a^k \) and \( x \in V_a^{k+1} \) hence \( x \in U_a^k \) and \( x \in U_a^{k+1} \) so \( x \in B_a \). Thus \( C'_a \subseteq V_a \).
We show that $U_a$ is open. Let $x \in U_a$. Then $x \in B_a$ a closed set so there exists an open set $M_x$ containing $x$ with $M_x \cap B_a = \emptyset$. Suppose $x \in E_a^k$ for some unique $k$. Then $x$ is in the interior of $E_a^k$ so there exists an open set $N_x$ with $x \in N_x \subseteq E_a^k$. Then $x \in M_x \cap N_x \cap U_a \subseteq U_a$. Suppose $x \in U_a^{k+1}$ some $k \in S$. Since $x$ is in the interior of $E_a^k \cup E_a^{k+1}$ choose an open set $N_x$ so that $x \in N_x \subseteq E_a^k \cup E_a^{k+1}$. Since $x \in U_a^{k+1}$ and $x \in U_a^k$ and $x \in B_a$ we have $x \in U_a^{k+1}$. Thus $x \in (U_a^k \cap U_a^{k+1}) \cap M_x \cap N_x \subseteq U_a$. Thus $U_a$ is open. A similar argument shows that $V_a$ is open. This completes the proof of Assertion 1.

**Assertion 2.** Order $\{B_a: \alpha < \eta\} \leq k + 1$.

**Proof.** Let $P_a = \bigcup_{j=0}^{N_a} (B_a^j \cap E_a^j)$. Then order $\{P_a: \alpha < \eta\} \leq k$ since order $\{B_a^j: j \in S, \alpha < \eta\} \leq k$. For $\alpha < \eta$ let $Q_a = \bigcup_{j=1}^{L_a} L_a^j$. Now order $\{Q_a: \alpha < \eta\} \leq 1$. Hence order $\{B_a: \alpha < \eta\} \leq \text{order } \{(P_a \cap Q_a): \alpha < \eta\} \leq k + 1$. This completes the proof of the theorem.

**Corollary.** Let $(X, \rho)$ be a metric space, $f: X \to [0, 1]$ a continuous function, $\sigma(x, y) = \rho(x, y) + |f(x) - f(y)|$, and let $d$ be $d_3$ or $d_5$. If $d(X, \rho) \leq k$ then $k \leq d(X, \sigma) \leq k + 1$.

**Theorem 2.** Let $(X, \rho)$ be a metric space and let $d$ be $d_3$ or $d_5$. Suppose $d(X, \rho) = r < n = \dim X$. Then for each $k, r \leq k \leq n$ there exists a topologically equivalent metric $\rho_k$ for $X$ such that $d(X, \rho_k) = k$.

**Proof.** Let $C_1, C'_1, C_2, C'_2, \ldots, C_n, C'_n$ be $n$ pairs of disjoint closed sets with the property that if for each $i = 1, \ldots, n$ $B_i$ is a closed set separating $C_i$ and $C'_i$, then $\bigcap_{i=1}^{n} B_i \neq \emptyset$. This is possible since $\dim X = n$. For each $i = 1, \ldots, n$ let $f_i: X \to [0, 1]$ such that $f$ is continuous, $f_i(C_i) = 0$ and $f_i(C'_i) = 1$. For each $i = 1, \ldots, n$ define $\sigma_i: X \times X \to \text{real numbers}$ by

$$\sigma_i(x, y) = \rho(x, y) + \sum_{j=1}^{i} |f_j(x) - f_j(y)|.$$ 

Now $\sigma_n(C_i, C'_i) \geq 1$ for all $i = 1, \ldots, n$ thus $d(X, \sigma_n) \geq n$. But by the above corollary $d(X, \sigma_{i+1}) \leq d(X, \sigma_i) + 1$. Thus all values $k, r \leq k \leq n$, are assumed and the theorem is proved.

**References**


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