EQUIVALENT METRICS GIVING DIFFERENT VALUES TO METRIC-DEPENDENT DIMENSION FUNCTIONS

J. C. NICHOLS

In [1] K. Nagami and J. H. Roberts introduced metric-dependent dimension functions $d_2$ and $d_3$ defined on the class of all metric spaces. All definitions are given below. The definition of $d_6$ is due to R. E. Hodel [5]. The following relations hold for all metric spaces $(X, \rho)$:

$$d_2(X, \rho) \leq d_3(X, \rho) \leq d_6(X, \rho) \leq \mu \text{dim}(X, \rho) \leq \text{dim } X,$$

where $\mu \text{dim}$ is metric dimension as defined by Katětov [3], and $\text{dim } X$ is covering dimension.

The following is a natural question. Suppose $d$ is $d_2$, $d_3$, $d_6$, or $\mu \text{dim}$; and suppose $d(X, \rho) = r < n = \text{dim } X$. Then for every $k$ ($r \leq k \leq n$) does there exist a topologically equivalent metric $\rho_k$ for $X$ such that $d(X, \rho_k) = k$? Roberts and Slaughter [2] answered this question in the affirmative when $d$ is $\mu \text{dim}$. Roberts [6] answered this question in the affirmative for all separable metric spaces when $d$ is $d_3$. This paper answers this question in the affirmative for all metric spaces when $d$ is $d_3$ or $d_6$. The question remains unanswered when $d$ is $d_2$.

In the following if $S$ is a set, $|S|$ will denote the cardinality of $S$.

Definition. Let $\eta$ be any ordinal number. A metric space $(X, \rho)$ is said to have property $P(|\eta|, k, \rho)$ if given any collection of pairs of closed sets indexed by $\eta$, $\mathcal{C} = \{(C_\alpha, C'_\alpha): \alpha < \eta\}$ such that there exists an $\epsilon > 0$ with $\rho(C_\alpha, C'_\alpha) \geq \epsilon$ for all $\alpha < \eta$ then there exists a collection of closed sets $\{B_\alpha: \alpha < \eta\}$ such that $B_\alpha$ separates $X$ between $C_\alpha$ and $C'_\alpha$ and order $\{B_\alpha: \alpha < \eta\} \leq k$.

Definition. $d_2(X, \rho)$ is the smallest integer $n$ such that $(X, \rho)$ has property $P(n+1, n, \rho)$.

Definition. $d_3(X, \rho)$ is the smallest integer $n$ such that $(X, \rho)$ has property $P(m, n, \rho)$ for every integer $m$.

Definition. $d_6(X, \rho)$ is the smallest integer $n$ such that $(X, \rho)$ has property $P(\mathbb{N}_0, n, \rho)$.

Definition. $\mu \text{dim } (X, \rho)$ is the smallest integer $n$ such that for each $\varepsilon > 0$ there exists an open cover $\mathcal{U}$ of $X$ with

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(i) \( \rho \)-mesh \( \mathcal{U} \leq \varepsilon \) and
(ii) order \( \mathcal{U} \leq n+1 \).

**Lemma.** Suppose \((X, \rho)\) is a metric space, \(f : X \to [0, 1]\) is a continuous function and \(\sigma(x, y) = \rho(x, y) + |f(x) - f(y)|\). Then \(\sigma\) is a metric on \(X\) topologically equivalent to \(\rho\). (See [4, p. 199].)

**Theorem 1.** Suppose \((X, \rho)\) is a metric space, \(f : X \to [0, 1]\) a continuous function and \(\sigma(x, y) = \rho(x, y) + |f(x) - f(y)|\).

1. If \(\eta\) is any ordinal number such that \(\mathbb{N}_0 \leq |\eta| \leq 2^{\mathbb{N}_0}\) and \((X, \rho)\) has property \(P(\eta, k, \rho)\) then \((X, \sigma)\) has property \(P(\eta, k+1, \sigma)\).
2. If \((X, \rho)\) has property \(P(m, k, \rho)\) for every integer \(m\) then \((X, \sigma)\) has property \(P(m, k+1, \sigma)\) for every integer \(m\).

**Remark.** To prove Theorem 2, the principle result of this paper, Theorem 1 is used only for the cases where \(\eta\) is countable or finite.

**Proof of Theorem 1.** Let \(\mathcal{E} = \{(C_\alpha, C'_\alpha) : \alpha < \eta\}\) be any collection of pairs of closed sets with \(|\eta| \geq 2^{\mathbb{N}_0}\) and with \(\sigma(C_\alpha, C'_\alpha) \geq \varepsilon\) for all \(\alpha < \eta\) and for some \(\varepsilon > 0\). Choose an integer \(N_0\) so that \(1/N_0 < \varepsilon/4\). Since \(|\eta| \leq 2^{\mathbb{N}_0}\) there exists a set \(T_0\) with \(T_0 \subseteq [0, 1/N_0)\) and with \(|T_0| = |\eta|\). We can assume that \(T_0 = \{t_\alpha : \alpha < \eta\}\) where if \(t_\alpha = t_\beta\) then \(\alpha = \beta\). Let \(S = \{0, 1, 2, \ldots, N_0\}\) and let \(t_0 = 0\) and \(t_{n+1} = 1\) for all \(\alpha < \eta\). For each \(i, 1 \leq i \leq N_0\) and each \(\alpha < \eta\) let \(t_\alpha = t_\alpha + (i - 1)/N_0\). Now for each \(i \in S\) and each \(\alpha < \eta\) define

\[
E_\alpha^i = \{x : \ t_\alpha^i \leq f(x) \leq t_{\alpha+1}^i\} \quad \text{and define}
\]

\[
D_\alpha^i = \{x : \rho((C_\alpha \cap E_\alpha^i), x) \geq \varepsilon/4\}.
\]

Then for \(i \in S\) and \(\alpha < \eta\) we have \(\rho(C_\alpha \cap E_\alpha^i, C_\alpha \cap E_\alpha^i) \geq 3\varepsilon/4\) because for \(x, y \in E_\alpha^i\) we have \(|f(x) - f(y)| \leq 1/N_0 \leq \varepsilon/4\). Thus \(D_\alpha^i \supseteq C_\alpha \cap E_\alpha^i\) for all \(i \in S\) and \(\alpha < \eta\). Now \(\mathcal{D} = \{(D_\alpha^i, C_\alpha \cap E_\alpha^i) : i \in S, \alpha < \eta\}\) is a collection of pairs of closed sets with \(\rho(D_\alpha^i, C_\alpha \cap E_\alpha^i) \geq \varepsilon/4\). In Case (1) where \(\mathbb{N}_0 \leq |\eta| \leq 2^{\mathbb{N}_0}\), we have \(|\mathcal{D}| = |\eta|\). In Case (2) we have \(|\eta| = m\) for some integer \(m\) and \(|\mathcal{D}| = mN_0\). Thus in either case our hypothesis guarantees the existence of a collection of closed sets \(\{B_\alpha^i : i \in S, \alpha < \eta\}\) with

1. order \(\{B_\alpha^i : i \in S, \alpha < \eta\} \leq k\)
2. \(X - B_\alpha^i = U_\alpha^i \cup V_\alpha^i\) where \(U_\alpha^i\) and \(V_\alpha^i\) are disjoint open sets and \(D_\alpha^i \subseteq V_\alpha^i\) and \(C_\alpha \cap E_\alpha^i \subseteq U_\alpha^i\).

For each \(\alpha < \eta\) we will modify the collection \(\{B_\alpha^i : i \in S\}\) to obtain a closed set \(B_\alpha\) separating \(X\) between \(C_\alpha\) and \(C'_\alpha\) using a variation of a method due to J. H. Roberts [7].

For each \(i \in S\) and \(\alpha < \eta\) define \(L_\alpha^i = \{x : f(x) = t_\alpha^i\}\). Notice that
$L^i_a = E^i_a \cap E^j_a$, for $i$ such that $1 \leq i \leq N_0$. For each $\alpha < \eta$ let $L^N_{\alpha+1} = \emptyset$ and define

$$B^\alpha = \bigcup_{j=0}^{N_0} \left[ (B^j_a \cap E^j_a) \cup (L^{j+1}_a \cap [(U^j_a - U^j_{a+1}) \cup (U^j_{a+1} - U^j_a)]) \right],$$

$$U^\alpha = \bigcup_{j=0}^{N_0} (U^j_a \cap E^j_a) - B^\alpha,$$

$$V^\alpha = \bigcup_{j=0}^{N_0} (V^j_a \cap E^j_a) - B^\alpha.$$

**Assertion 1.** For each $\alpha < \eta$ $B^\alpha$ is a closed set separating $X$ between $C^\alpha$ and $C^{\alpha'}$.

**Proof.** First we show that $B^\alpha$ is closed. Let $H^\alpha = \bigcup_{j=0}^{N_0} (B^j_a \cap E^j_a)$ and let $G^\alpha = \bigcup_{j=0}^{N_0} (L^{j+1}_a \cap [(U^j_a - U^j_{a+1}) \cup (U^j_{a+1} - U^j_a)])$. It suffices to show that $G^\alpha \subseteq B^\alpha$ since $H^\alpha$ is closed. If $x$ is a limit point of $G^\alpha$ then there exists some $k \in S$ such that $x$ is a limit point of $L^{k+1}_a \cap [(U^k_a - U^k_{a+1}) \cup (U^k_{a+1} - U^k_a)]$. We may assume then that $x$ is a limit point of $(U^k_a - U^k_{a+1})$ hence a limit point of $U^k_a$. But $X - B^k_a = U^k_a \cup V^k_a$ where $U^k_a \cap V^k_a = \emptyset$. Thus either $x \in U^k_a$ or $x \in B^k_a$ and in either case $x \in B^\alpha$ so $B^\alpha$ is closed.

Next we show that $X - B^\alpha = U^\alpha \cup V^\alpha$. If $x \in X$, there exists $k \in S$ such that $x \in E^k_a$, since $\bigcup_{j=0}^{N_0} E^j_a = X$. If $x \in B^\alpha$ then surely $x \in B^k_a \cap E^k_a$.

But $X - B^k_a = U^k_a \cup V^k_a$ so $x$ is in one of $U^k_a$ or $V^k_a$ hence one of $U^\alpha$ or $V^\alpha$.

We show that $U^\alpha \cap V^\alpha = \emptyset$. If $x \in U^\alpha$ then either $x \in E^\alpha_k$ for exactly one $k \in S$ or $x \in (E^\alpha_k \cap E^{\alpha+1}_a)$ for exactly one $k \in S$. In the first case since $x \in U^\alpha_k$ we have $x \in V^\alpha_k$ hence $x \in V^\alpha$. In the second case we can suppose that $x \in U^\alpha_k$. The only possibility to have $x \in V^\alpha$ is to have $x \in V^\alpha_{k+1}$. But then $x \in U^\alpha_{k+1}$ hence $x \in (U^\alpha_k - U^\alpha_{k+1}) \cap (E^\alpha_k \cap E^{\alpha+1}_a)$. Thus $x \in B^\alpha$ and $x \in U^\alpha$ so we conclude that $U^\alpha \cap V^\alpha = \emptyset$.

To show that $C^\alpha \subseteq U^\alpha$ we first show that $C^\alpha \cap \partial B^\alpha = \emptyset$. Let $x \in C^\alpha$ and suppose that $x \in E^\alpha_k$ for exactly one $k$. Then $x \in U^\alpha_k$ for any $j \in S$. Now $(C^\alpha \cap E^\alpha_k) \cap B^\alpha = \emptyset$ because $X - B^\alpha = U^\alpha \cup V^\alpha$ where $(C^\alpha \cap E^\alpha_k) \subseteq U^\alpha_k$. So in this case $x \in B^\alpha$. If $x \in E^\alpha_k \cap E^{\alpha+1}_a$ for some $k \in S$ then $x \in U^\alpha_k$ and $x \in U^\alpha_{k+1}$. Thus $x \in B^\alpha$ and $C^\alpha \cap \partial B^\alpha = \emptyset$. Since $C^\alpha \subseteq \bigcup_{j=0}^{N_0} (U^j_a \cap E^j_a)$ and $C^\alpha \cap B^\alpha = \emptyset$ we conclude that $C^\alpha \subseteq U^\alpha$.

From the definition of $D^j_a$ it is clear that $C^\alpha \subseteq \bigcup_{j=0}^{N_0} D^j_a$ but $D^j_a \subseteq V^j_a$ so $C^\alpha \subseteq \bigcup_{j=0}^{N_0} V^j_a$. Thus if we show that $C^\alpha \cap B^\alpha = \emptyset$ we can conclude that $C^\alpha \subseteq V^\alpha$. Let $x \in C^\alpha$ and let $x \in E^\alpha_k$ for exactly one $k$. Then $x \in V^\alpha_k$ so $x \in B^\alpha_k$ hence $x \in B^\alpha$. If $x$ is in $E^\alpha_k \cap E^{\alpha+1}_a$ then $x \in V^\alpha_k$ and $x \in V^\alpha_{k+1}$ hence $x \in U^\alpha_k$ and $x \in U^\alpha_{k+1}$ so $x \in B^\alpha$. Thus $C^\alpha \subseteq V^\alpha$. 


We show that $U_a$ is open. Let $x \in U_a$. Then $x \in \mathcal{B}_a$, a closed set, so there exists an open set $M_x$ containing $x$ with $M_x \cap \mathcal{B}_a = \emptyset$. Suppose $x \in E^k_a$ for some unique $k$. Then $x$ is in the interior of $E^k_a$ so there exists an open set $N_x$ with $x \in N_x \subseteq E^k_a$. Then $x \in M_x \cap N_x \cap U^k_a \subseteq U_a$. Suppose $x \in U^{k+1}_a$ some $k \in S$. Since $x$ is in the interior of $E^k_a \cup E^{k+1}_a$ choose an open set $N_x$ so that $x \in N_x \subseteq E^k_a \cup E^{k+1}_a$. Since $x \in U^{k+1}_a$ and $x \in U^k_a$ and $x \in \mathcal{B}_a$ we have $x \in U^k_a$. Thus $x \in (U^k_a \cap U^{k+1}_a) \cap M_x \cap N_x \subseteq U_a$. Thus $U_a$ is open. A similar argument shows that $V_a$ is open. This completes the proof of Assertion 1.

**Assertion 2.** Order \( \{ B_a : \alpha < \eta \} \leq k + 1 \).

**Proof.** Let $P_a = \bigcup_{j=0}^{N_a} (B^j_a \cap E^j_a)$. Then order \( \{ P_a : \alpha < \eta \} \leq k \) since order \( \{ B^j_a : j \in S, \alpha < \eta \} \leq k \). For $\alpha < \eta$ let $Q_a = \bigcup_{j=0}^{L_a} L_a$. Now order \( \{ Q_a : \alpha < \eta \} \leq 1 \). Hence order \( \{ B_a : \alpha < \eta \} \) order \( \{ (P_a \cap Q_a) : \alpha < \eta \} \leq k + 1 \). This completes the proof of the theorem.

**Corollary.** Let $(X, \rho)$ be a metric space, $f : X \to [0, 1]$ a continuous function, $\sigma(x, y) = \rho(x, y) + \left| f(x) - f(y) \right|$, and let $d$ be $d_3$ or $d_6$. If $d(X, \rho) \leq k$ then $k \leq d(X, \sigma) \leq k + 1$.

**Theorem 2.** Let $(X, \rho)$ be a metric space and let $d$ be $d_3$ or $d_6$. Suppose $d(X, \rho) = r < n = \dim X$. Then for each $k$, $r \leq k \leq n$ there exists a topologically equivalent metric $\rho_k$ for $X$ such that $d(X, \rho_k) = k$.

**Proof.** Let $C_1, C'_1; C_2, C'_2; \ldots, C_n, C'_n$ be $n$ pairs of disjoint closed sets with the property that if for each $i = 1, \ldots, n$ $B_i$ is a closed set separating $C_i$ and $C'_i$, then $\bigcap_{i=1}^{n} B_i \neq \emptyset$. This is possible since dim $X = n$. For each $i = 1, \ldots, n$ let $f_i : X \to [0, 1]$ such that $f$ is continuous, $f_i(C_i) = 0$ and $f_i(C'_i) = 1$. For each $i = 1, \ldots, n$ define $\sigma_i : X \times X \to \mathbb{R}$ by

$$\sigma_i(x, y) = \rho(x, y) + \sum_{j=1}^{i} \left| f_j(x) - f_j(y) \right| .$$

Now $\sigma_i(C_i, C'_i) \geq 1$ for all $i = 1, \ldots, n$ thus $d(X, \sigma_n) \geq n$. But by the above corollary $d(X, \sigma_{i+1}) \leq d(X, \sigma_i) + 1$. Thus all values $k, r \leq k \leq n$, are assumed and the theorem is proved.

**References**


Duke University