FINITE GROUPS WITH SOLVABLE MAXIMAL SUBGROUPS

J. W. RANDOLPH

R. Baer [1] has established relationships between maximal subgroups and minimal normal subgroups. We note here some applications of these relationships if a group has a solvable maximal subgroup.

All groups considered are finite. If H is a subgroup of a group G, we denote the centralizer of H in G, the normalizer of H in G and the largest normal subgroup of G in H by $C(H)$, $N(H)$ and $\mathrm{core}(H)$, respectively. The identity of G will be denoted by 1, and the order of G by $|G|$.

THEOREM. *Let G be a group containing a solvable maximal subgroup S and a non-Abelian minimal normal subgroup M. Then*

(1) *$C(M)$ and $\mathrm{core}(S)$ are both equal to the intersection of all solvable maximal subgroups of G,*

(2) *S is not normal in G,*

(3) *M is the only non-Abelian minimal normal subgroup of G,*

(4) $\mathrm{core}(S)=1$ *if and only if M is the only minimal normal subgroup of G,*

(5) *if S is nilpotent then the intersection of all solvable maximal subgroups of G is equal to the intersection of all nilpotent maximal subgroups of G, the 2-Sylow subgroup P of S is a non-Abelian 2-Sylow subgroup of G, and $N(P)=S$, and*

(6) *if S is nilpotent and* $\mathrm{core}(S)=1$ *then S is a non-Abelian 2-Sylow subgroup of G.*

PROOF.

LEMMA [1, pp. 119–120]. *Let G be a group containing a maximal subgroup S such that* $\mathrm{core}(S)=1$. *Then*

(a) *if H is a normal subgroup of G and $H\neq 1$, then $C(H)\cap S=1$ and $C(H)=1$ or $C(H)$ is a minimal normal subgroup of G,*

(b) *there exist at most two different minimal normal subgroups of G, and*

(c) *if A and B are two different minimal normal subgroups of G, then A, B and $AB\cap S$ are isomorphic.*

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Let $G^*=G/\text{core}(S)$, $S^*=S/\text{core}(S)$ and $M^*=M\ \text{core}(S)/\text{core}(S)$. Since solvable minimal normal subgroups of finite groups are Abelian [**3**, p. 74], $M\not\subseteq S$, so M^* is a minimal normal subgroup of G^*. Also $\text{core}(S^*)=1^*$, the identity of G^*, as noted by Baer [**1**, p. 123]. Moreover $M\cap\text{core}(S)=1$, whence M and M^* are isomorphic and $\text{core}(S)\subseteq C(M)$. If M^* and N^* were distinct minimal normal subgroups of G^*, then by the lemma, M^* and $M^*N^*\cap S^*$ would be isomorphic; but S is solvable so then M^* (and hence also M) would be solvable, a contradiction. So M^* is the only minimal normal subgroup of G^*. If $C(M^*)\neq 1^*$, then by the lemma $C(M^*)$ is a minimal normal subgroup of G^*, so $C(M^*)=M^*$ and M^* is Abelian, a contradiction. Thus $C(M^*)=1^*$. But $C(M^*)=C(M)\text{core}(S)/\text{core}(S)$, and consequently $C(M)\subseteq\text{core}(S)\subseteq S$. Let R denote the intersection of all solvable maximal subgroups of G. Then $C(M)\subseteq R$. But R is a solvable normal subgroup of G, so $M\cap R=1$ and $R\subseteq C(M)$, proving (1).

If S were normal in G, G^* would have prime order and M^* (and hence also M) would be Abelian, a contradiction, proving (2).

If G had a non-Abelian minimal normal subgroup N such that $N\neq M$, then by (1), $C(M)=C(N)$. But since M and N are both minimal normal and $M\neq N$, $M\cap N=1$, so $M\subseteq C(N)=C(M)$ and M is Abelian, a contradiction, and (3) is proved. The truth of (4) is evident.

For the remainder of the proof, suppose S is nilpotent. The first assertion of (5) follows from proving in the same manner as for (1) that $C(M)$ is the intersection of all nilpotent maximal subgroups of G. If a group G has a nilpotent maximal subgroup whose 2-Sylow subgroup is Abelian, then G is solvable [**2**, p. 281]. But M is not Abelian, so the 2-Sylow subgroup P of S is not Abelian.

Suppose additionally that $\text{core}(S)=1$. Then S must be a Hall subgroup of G, since otherwise [**3**, p. 392] G would have an Abelian minimal normal subgroup, contradicting (4) of our theorem. Assume S is not a Sylow subgroup of G. Let Q be any Sylow subgroup of S with $Q\neq 1$. Since S is nilpotent, Q is normal in S. Thus $S\subseteq N(Q)$, and by the maximality of S, $S=N(Q)$ or Q is normal in G. But $\text{core}(S)=1$ so $S=N(Q)$. Therefore [**3**, p. 377] S has a normal complement, say K, in G. Since S is nilpotent, G/K must be solvable. But S is a Hall subgroup of G and S has even order, so K has odd order, whence K is solvable, by the Feit-Thompson theorem. But then G is solvable, a contradiction, proving (6).

If $\text{core}(S)\neq 1$, consider S^* which is a nilpotent maximal subgroup of G^* and satisfies $\text{core}(S^*)=1^*$. Applying (6) to S^*, we see that S^* is a non-Abelian 2-Sylow subgroup of G^*. Hence [**3**, p. 134] S^*

$=J\ \text{core}(S)/\text{core}(S)$ for some non-Abelian 2-Sylow subgroup J of G. Then $|S^*|=|J/J\cap\text{core}(S)|$. But $|S|=|S^*|\,|\text{core}(S)|$, so $|S| =|J|\,|\text{core}(S)|/|J\cap\text{core}(S)|$. It follows that P, the 2-Sylow subgroup of S, is a non-Abelian 2-Sylow subgroup of G. Finally, as before, $S=N(P)$ or P is normal in G; in the latter case $P=J$, $J\subseteq\text{core}(S)$ and S^* is Abelian, a contradiction. This completes the proof of (5) and the theorem.

REMARK. It follows that if a non-Abelian simple group G has a nilpotent maximal subgroup S, then S is a non-Abelian 2-Sylow subgroup of G. This fact is an immediate consequence of Thompson's normal p-complement theorem in J. Algebra **1** (1964), 43–46.

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WEST VIRGINIA UNIVERSITY