A NOTE ON A THEOREM OF WEYL
KENNETH K. WARNER

In 1909, H. Weyl [7] proved that for a selfadjoint operator in Hilbert space, perturbation by addition of a selfadjoint compact operator leaves an "essential" part of the spectrum invariant (namely, the limit points of the spectrum and the points of infinite multiplicity). Coburn [1] has given an extension of this theorem to hyponormal operators. Nieto [3] has obtained a generalization of the theorem to a class of operators in a Banach space. The purpose of this note is to give another interpretation of the theorem of Weyl. Hyponormal and symmetrizable operators satisfy this theorem of Weyl-type. However, these results are apparently weaker than those of Coburn and Nieto. The connection with Coburn's theorem is established.

Let $T$ be a bounded linear operator on a Banach space $X$. Define the nullity, $n(T)$, as the dimension of $N(T) = \{x: Tx = 0\}$ and the defect, $d(T)$, as the dimension of $X/R(T)$, where $R(T)$ is the range of $T$. The index of $T$ is given by $k(T) = d(T) - n(T)$. Let $\mathfrak{B}(X)$ denote the bounded linear operators on $X$, $\mathfrak{C}(X)$ the compact linear operators. Let $\sigma(T)$ be the spectrum of the operator $T$. After Coburn [1], we define the Weyl spectrum by

$$\omega(T) = \bigcap_{c \in \mathfrak{C}(X)} \sigma(T + C)$$

Schechter [4] has shown that the complement of $\omega(T)$, $\mathcal{C}\omega(T) = \{\lambda: k(\lambda - T) = 0$, and $d(\lambda - T) < \infty$ and $n(\lambda - T) < \infty\}$. Let $\alpha(T)$ denote the ascent of $T$; that is, the smallest nonnegative such that the null spaces $N(T^n)$ are equal if $n \geq \alpha(T)$. The descent $\delta(T)$ is the smallest integer such that the ranges $R(T^n)$ are equal if $n \geq \delta(T)$. Finally, if $\lambda_0$ is an isolated point of $\sigma(T)$ that is a pole of the resolvent operator $(\lambda - T)^{-1}$, then $\lambda_0$ is a pole of the resolvent of finite rank if the spectral projection associated with $\{\lambda_0\}$ has finite dimensional range (see Taylor [5], [6]).

The following gives a general condition for a theorem of Weyl type to hold.

**Theorem.** If the ascent of $\lambda - T$ is finite for a dense set of points $\lambda$, then $\omega(T)$ consist of $\sigma(T)$ except for poles of the resolvent of finite rank.

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Proof. Since $\omega(T)$ is the intersection of closed sets, then its complement $C \omega(T)$ is open. If $\lambda \in C \omega(T)$, then $n(\lambda - T) = d(\lambda - T) < \infty$. If $\lambda_0 \in C \omega(T) \cap \sigma(T)$, then there is an open connected set $S$ about $\lambda_0$ such that $n(\lambda - T) = d(\lambda - T)$ for $\lambda \in S$. For each $\mu \in S$ with finite ascent, we conclude, by Theorem 4.5 of Taylor [5], that $\alpha(\mu - T) = \delta(\mu - T)$. Now, Theorem 9.4 of Taylor [5] implies that $\mu$ is an isolated point of $\sigma(T)$. Thus, $S \cap \rho(T) \neq \emptyset$, where $\rho(T)$ is the resolvent set for $T$. Theorem 9.6 of Taylor [5] implies that $S \cap \sigma(T)$ consists of isolated points of $\sigma(T)$ that correspond to poles of the resolvent, $(\lambda - T)^{-1}$, of finite rank. Thus $\sigma(T) \cap C \omega(T)$ consists of isolated points such that each point is a pole of finite rank of the resolvent.

If $\lambda$ is an isolated point of $\sigma(T)$ that is a pole of the resolvent of finite rank, then $n(\lambda - T) = d(\lambda - T) < \infty$ by Theorem 9.2 of Taylor [5]. Consequently, $\lambda \in \omega(T)$ by Schechter's characterization of $C \omega(T)$.

Corollary 1. If the spectrum of $T$ is a nowhere dense set, then $\omega(T)$ consists of $\sigma(T)$ except for poles of the resolvent of finite rank.

The result analogous to Coburn's [1] generalization to hyponormal operators of the Weyl theorem depends on the following lemma. An operator is said to be hyponormal if $A^*A \geq AA^*$.

Lemma 1. If $A$ is a hyponormal operator on a Hilbert space, then $A$ has ascent 0 or 1.

Proof. Let $x \in N(A^2)$. Then
\[
((A^*A)^2 x, x) = (A A^* Ax, Ax) \leq (A^* A Ax, Ax) = 0,
\]
since $A^*A \geq AA^*$. Thus $(A^*A^2 x, x) = 0$, which implies that
\[
(A^* A x, A^* A x) = \|A^* A x\|^2 = 0.
\]
Hence $A^* A x = 0$. But
\[
0 = (A^* A x, x) = (Ax, Ax) = \|Ax\|^2,
\]
implies that $x \in N(A)$, or $N(A^2) \subseteq N(A)$. Thus the ascent is 0 or 1.

Corollary 2. If $A$ is hyponormal, then $\omega(A)$ consists of $\sigma(A)$ except for poles of the resolvent of finite rank.

Proof. Since $A$ is hyponormal, $\lambda - A$ is hyponormal for all $\lambda$. Hence, $\alpha(\lambda - A)$ is finite for all $\lambda$. Theorem 1 now gives the desired conclusion.

The connection between Corollary 2 and Coburn's generalization
of the theorem of Weyl is given in Lemma 2. A point $\lambda \in \sigma(T)$ has finite multiplicity if $n(\lambda - T) < \infty$.

**Lemma 2.** Let $T$ be a hyponormal operator. An isolated point of $\sigma(T)$ of finite multiplicity is a pole of the resolvent of finite rank.

**Proof.** In the proof of Theorem 3.1 of Coburn [1], it is shown that if $\lambda$ is an isolated point of finite multiplicity of a hyponormal operator, then $k(\lambda - T) = 0$. Now Theorem 9.6 of Taylor [5] implies that $\lambda$ is a pole of the resolvent of finite rank.

Let $\phi$ be a continuous isomorphism of a Banach space $X$ into a Hilbert space $H$ such that $\phi(X)$ is dense in $H$. A bounded linear operator $T$ on $X$ is said to be symmetrizable if the induced operator $\hat{T}\phi(x) = \phi(Tx)$ is symmetric with domain $\phi(X)$. The result analogous to Nieto's [2] theorem is immediate after the lemma.

**Lemma 3.** Let $T$ be a linear operator on $X$. Suppose $T$ is symmetrizable. Then if $\text{Im}\lambda \neq 0$, then $\alpha(\lambda - T) = 0$ or 1.

**Proof.** Denote the norm in $H$ by $\| \cdot \|_H$. For a symmetric operator $S$ on $H$, Kato [2], p. 270, shows that for $u$ in the domain of $S$

$$\| (\lambda - S)u \|_H^2 = \| (\text{Re} \lambda - S)u \|_H^2 + (\text{Im} \lambda)^2 \| u \|_H^2.$$  

Consequently, if $(\lambda - T)^2x = 0$, let $u = \phi((\lambda - T)x)$ and $S = \hat{T}$.

Then $(\lambda - S)u = \phi((\lambda - T)x) = 0$. The above equality implies $\text{Im} \lambda \neq 0$, that $\| \phi[(\lambda - T)x] \|_H = 0$, or $(\lambda - T)x = 0$, since $\phi$ is an isomorphism.

**Corollary 3.** If $T$ is symmetrizable, then $\omega(T)$ consists of $\sigma(T)$ except for poles of the resolvent of finite rank.

**References**


**California State College, Long Beach**