ON SUBPARACOMPACT SPACES[1]

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1. **Introduction.** This paper is concerned with establishing equivalence between several classes of topological spaces. The main result is Theorem 1.2, for which a proof is given in §2. Throughout this paper the set of positive integers will be denoted by Z^+.

In [2] Arhangel'skiĭ introduces the class of σ-paracompact spaces. Following Arhangel'skiĭ, a space X is called *σ-paracompact* if for any open covering $\mathfrak{U}$ of X there is a sequence $\{\mathfrak{U}_n\}_{n=1}^{\infty}$ of open coverings of X such that if $x \in X$ there is $m(x) \in Z^+$ and some set $U \in \mathfrak{U}$ with $\mathrm{St}(x, \mathfrak{U}_{m(x)}) \subset U$.

The class of σ-paracompact spaces was studied recently by Čoban in [5], and the author and R. Stoltenberg in [4]. The following proposition was proved independently in both papers.

PROPOSITION 1.1. *If X is a topological space with the property that every open cover of X has a σ-discrete closed refinement, then X is σ-paracompact.*

Spaces with the property that every open cover has a σ-discrete closed refinement are called *F_σ-screenable* by McAuley in [9].

We now state a much stronger result than Proposition 1.1, which says that σ-paracompactness is actually equivalent to several conditions on a topological space and, in particular, to the F_σ-screenable condition.

THEOREM 1.2. *For a topological space X the following conditions are equivalent:*
   (a) *X is σ-paracompact.*
   (b) *Every open cover of X has a σ-discrete closed refinement.*
   (c) *Every open cover of X has a σ-locally-finite closed refinement.*
   (d) *Every open cover of X has a σ-closure-preserving closed refinement.*

In view of the above theorem and because it appears that the terms σ-paracompact and F_σ-screenable are not very satisfactory, we make the following definition:

DEFINITION 1.3. A topological space X is called *subparacompact* if it satisfies any one of the conditions (a) through (d) stated in Theorem 1.2.

Clearly the class of subparacompact spaces includes the class of developable[2] spaces. In fact it is known that semimetric spaces [9] and semistratifiable spaces [6] satisfy (b) and hence are subparacompact.

A type of topological space which naturally possesses property (c) is a space with a σ-locally-finite closed network.[3] Spaces with a σ-locally-finite network are called σ-spaces by Okuyama in [12] and [13]. Assuming regularity, Siwiec and Nagata have shown in [14] that the class of σ-spaces is equivalent to the class of spaces with a σ-discrete network and to the class of spaces with a σ-closure-preserving network.

REMARK 1.4. If X is a regular space we need not require that the refinements as given in (b), (c), or (d) of Theorem 1.2 consist of closed sets.

2. **Proof of Theorem** 1.2. It suffices to prove (a)$\rightarrow$(b) and (d)$\rightarrow$(b), since it has been established by Proposition 1.1 that (b)$\rightarrow$(a) and it is obvious that (b)$\rightarrow$(c)$\rightarrow$(d).

(a)$\rightarrow$(b). Assume X satisfies condition (a) of Theorem 1.2.

LEMMA 2.1. *If $\mathfrak{U}$ is any open cover of X there is a sequence $\{\mathfrak{U}_n\}_{n=1}^{\infty}$ of open covers of X satisfying*:

(1) $\mathfrak{U}_1 = \mathfrak{U}$ *and $\mathfrak{U}_{n+1}$ refines $\mathfrak{U}_n$ for each $n \in Z^+$.*

(2) *Given $n \in Z^+$, $x \in X$ there is an integer $m \in Z^+$ (m depends on n and x) and some $U_n \in \mathfrak{U}_n$ such that $\mathrm{St}(x, \mathfrak{U}_m) \subset U_n$.*

PROOF. By induction, for each $k \in Z^+$, we can find a sequence $\{\mathfrak{U}_n(k)\}_{n=k}^{\infty}$ of open covers of X such that:

(i) $\mathfrak{U}_1(1) = \mathfrak{U}$ and $\mathfrak{U}_{k+1}(k+1) = \mathfrak{U}_{k+1}(k)$.

(ii) $\mathfrak{U}_{n+1}(k)$ refines $\mathfrak{U}_n(k)$ for each $n \geq k$.

(iii) $\mathfrak{U}_n(k+1)$ refines $\mathfrak{U}_n(k)$ for each $n \geq k+1$.

(iv) Given $x \in X$ there is an integer $m \in Z^+$ (m depends on k and x) such that $\mathrm{St}(x, \mathfrak{U}_m(k)) \subset U_k$ for some $U_k \in \mathfrak{U}_k(k)$.

Let $\mathfrak{U}_k = \mathfrak{U}_k(k)$; it follows that $\{\mathfrak{U}_k\}_{k=1}^{\infty}$ is a sequence of open covers with the desired properties.

Let $\mathfrak{U} = \{U_\alpha : \alpha \in \Gamma\}$ be an open cover of X with Γ well-ordered. Then there is a sequence $\{\mathfrak{U}_n\}_{n=1}^{\infty}$ of open covers of X satisfying (1) and (2) of Lemma 2.1.

For each $x \in X$ let

$$\Gamma_x = \{\alpha \in \Gamma \colon \mathrm{St}(x, \mathfrak{U}_n) \subset U_\alpha \text{ for some } n \in Z^+\},$$

and define $\alpha(x)$ to be the least element of Γ_x.

For $\alpha \in \Gamma$ and $n \in Z^+$ let

$$P_n(\alpha) = \{z \in X \colon \mathrm{St}(z, \mathfrak{U}_n) \subset U_\alpha \text{ and } \alpha = \alpha(z)\}.$$

Note that $\{P_n(\alpha) \colon \alpha \in \Gamma\}$ is a disjoint collection for each $n \in Z^+$, and the collection $\{P_n(\alpha) \colon \alpha \in \Gamma, n \in Z^+\}$ covers X and refines $\mathfrak{U}$.

Now if $n, m \in Z^+$, with $m \geq n$, we define

$$P_{n,m}(\alpha) = \{z \in P_n(\alpha) \colon \mathrm{St}(z, \mathfrak{U}_m) \subset U_n \text{ for some } U_n \in \mathfrak{U}_n\}.$$

It follows that $\bigcup_{m=n}^\infty P_{n,m}(\alpha) = P_n(\alpha)$, so the collection

$$\mathcal{P} = \{P_{n,m}(\alpha) \colon n, m \in Z^+, m \geq n, \alpha \in \Gamma\}$$

covers X and refines $\mathfrak{U}$. We will show that for a fixed n, $m \in Z^+$, $m \geq n$, the family $\mathcal{P}_{n,m} = \{P_{n,m}(\alpha) \colon \alpha \in \Gamma\}$ is a discrete collection of sets in X.

Let $P_{n,m}(\alpha)$ be a nonempty element of $\mathcal{P}_{n,m}$ and z an arbitrary element of $P_{n,m}(\alpha)$. Suppose there is some $\beta \in \Gamma$, $\beta \neq \alpha$, such that $\mathrm{St}(z, \mathfrak{U}_m) \cap P_{n,m}(\beta) \neq \varnothing$. Let $y \in \mathrm{St}(z, \mathfrak{U}_m) \cap P_{n,m}(\beta)$. Now $y \in P_{n,m}(\beta)$ implies that $\mathrm{St}(y, \mathfrak{U}_m) \subset U_n$ for some $U_n \in \mathfrak{U}_n$ and $y \in \mathrm{St}(z, \mathfrak{U}_m)$ implies that $z \in \mathrm{St}(y, \mathfrak{U}_m) \subset U_n$. Thus $\mathrm{St}(y, \mathfrak{U}_m) \subset \mathrm{St}(z, \mathfrak{U}_n) \subset U_\alpha$, so $\alpha \in \Gamma_y$. Hence $\beta < \alpha$ since $\beta = \alpha(y)$ and $\alpha(y)$ is the least element of Γ_y. Also $z \in P_{n,m}(\alpha)$ implies that $\mathrm{St}(z, \mathfrak{U}_m) \subset U_n'$ for some $U_n' \in \mathfrak{U}_n$. But $y \in \mathrm{St}(z, \mathfrak{U}_m) \subset U_n'$ implies that $\mathrm{St}(z, \mathfrak{U}_m) \subset \mathrm{St}(y, \mathfrak{U}_n) \subset U_\beta$, so $\beta \in \Gamma_z$. Hence $\alpha(z) = \alpha < \beta$ and we have a contradiction.

Since z was an arbitrary element of $P_{n,m}(\alpha)$ we have shown that $\mathrm{St}(P_{n,m}(\alpha), \mathfrak{U}_m) \cap P_{n,m}(\beta) = \varnothing$ whenever $\beta \neq \alpha$.

To complete the proof that $\mathcal{P}_{n,m}$ is discrete we show for each $x \in X$ that there is a neighborhood of x which intersects at most one element of $\mathcal{P}_{n,m}$. Let $P_{n,m} = \bigcup \{P_{n,m}(\alpha) \colon \alpha \in \Gamma\}$. If $x \in X - \overline{P}_{n,m}$ we are through, so suppose $x \in \overline{P}_{n,m}$. Now

$$P_{n,m} \subset X - \bigcup \{U \in \mathfrak{U}_m \colon U \cap P_{n,m} = \varnothing\}$$

$$\subset \bigcup \{\mathrm{St}(P_{n,m}(\alpha), \mathfrak{U}_m) \colon \alpha \in \Gamma\} = \mathrm{St}(P_{n,m}, \mathfrak{U}_m)$$

and $X - \bigcup\{U \in \mathfrak{U}_m \colon U \cap P_{n,m} = \varnothing\}$ is a closed set. Thus $\overline{P}_{n,m} \subset \mathrm{St}(P_{n,m}, \mathfrak{U}_m)$. So $x \in \mathrm{St}(P_{n,m}, \mathfrak{U}_m)$ and there must be some $\beta \in \Gamma$ such that $x \in \mathrm{St}(P_{n,m}(\beta), \mathfrak{U}_m)$. We have shown above that $\mathrm{St}(P_{n,m}(\beta), \mathfrak{U}_m)$ does not intersect any $P_{n,m}(\alpha)$ for $\alpha \neq \beta$. Hence $\mathrm{St}(P_{n,m}(\beta), \mathfrak{U}_m)$ is a neighborhood of x which intersects only one element of $\mathcal{P}_{n,m}$. It follows that $\mathcal{P} = \bigcup_{n=1}^\infty \bigcup_{m=n}^\infty \mathcal{P}_{n,m}$ is a σ-discrete refinement of $\mathfrak{U}$.

To complete the proof that (a)$\rightarrow$(b) note that if $P_{n,m}(\alpha) \in \mathcal{P}_{n,m}$,

then $\mathrm{Cl}(P_{n,m}(\alpha))\subset\overline{P}_{n,m}\subset\mathsf{U}\{\mathrm{St}(P_{n,m}(\beta),\,\mathfrak{U}_m):\beta\in\Gamma\}$ and $\mathrm{Cl}(P_{n,m}(\alpha))$ $\cap\mathrm{St}(P_{n,m}(\beta),\,\mathfrak{U}_m)=\varnothing$ for $\beta\neq\alpha$. Thus $\mathrm{Cl}(P_{n,m}(\alpha))\subset\mathrm{St}(P_{n,m}(\alpha),\,\mathfrak{U}_m)$ $\subset\mathrm{St}(P_{n,m}(\alpha),\,\mathfrak{U}_n)\subset U_\alpha$. We have shown that

$$\mathfrak{P}' = \big\{\mathrm{Cl}(P_{n,m}(\alpha)): P_{n,m}(\alpha) \in \mathfrak{P}\big\}$$

is a closed refinement of $\mathfrak{U}$. Since $\mathfrak{P}$ is σ-discrete, $\mathfrak{P}'$ is a σ-discrete closed refinement.

(d)$\rightarrow$(b). Assume every open cover of X has a σ-closure-preserving closed refinement.

LEMMA 2.2. *For each $n\in Z^+$, let $\mathfrak{U}(n) = \big\{ U_\alpha(n): \alpha\in\Gamma\big\}$ be an open cover of X such that $U_\alpha(n+1)\subset U_\alpha(n)$ for all $\alpha\in\Gamma$. Then there is a sequence $\big\{\mathfrak{P}(n)\big\}_{n=1}^{\infty}$ of closed coverings of X such that, for each $n\in Z^+$, $\mathfrak{P}(n) = \mathsf{U}_{m=1}^{\infty}\,\mathfrak{P}_m(n)$ and the following conditions are satisfied:*

(1) *$\mathfrak{P}_m(n) = \big\{P_{\alpha,m}(n): \alpha\in\Gamma\big\}$ and is closure-preserving for each $m\in Z^+$.*

(2) *$P_{\alpha,m}(n)\subset U_\alpha(n)$ for each $\alpha\in\Gamma$, $m\in Z^+$.*

(3) *$P_{\alpha,m}(n)\subset P_{\alpha,m+1}(n)$ for each $\alpha\in\Gamma$, $m\in Z^+$.*

(4) *$P_{\alpha,m}(n+1)\subset P_{\alpha,m}(n)$ for each $\alpha\in\Gamma$, $m\in Z^+$.*

PROOF. Each $\mathfrak{U}(n)$ has a closed refinement $\mathfrak{B}(n) = \mathsf{U}_{m=1}^{\infty}\,\mathfrak{B}_m(n)$ where each $\mathfrak{B}_m(n)$ is closure-preserving and $\mathfrak{B}_m(n)\subset\mathfrak{B}_{m+1}(n)$. Let $m,\,n\in Z^+$ and $\alpha\in\Gamma$. If $m<n$ let $P_{\alpha,m}(n) = \varnothing$. If $m\geqq n$ define

$$P_{\alpha,m}(n) = \mathsf{U}\,\big\{B \in \mathfrak{B}_m(k): B \subset U_\alpha(n),\, k \in Z^+,\, n \leqq k \leqq m\big\}.$$

Let $\mathfrak{P}_m(n) = \big\{P_{\alpha,m}(n): \alpha\in\Gamma\big\}$ and $\mathfrak{P}(n) = \mathsf{U}_{m=1}^{\infty}\,\mathfrak{P}_m(n)$. It is easily verified that $\big\{\mathfrak{P}(n)\big\}_{n=1}^{\infty}$ is the required sequence of closed covers.

Let $\mathfrak{U} = \big\{U_\alpha: \alpha\in\Gamma\big\}$ be an open cover of X with Γ well ordered. We will construct a σ-discrete closed refinement of $\mathfrak{U}$. For notational purposes let $U_\alpha(1,\,n) = U_\alpha$ and $\mathfrak{U}(1,\,n) = \big\{U_\alpha(1,\,n): \alpha\in\Gamma\big\}$ for each $n\in Z^+$. There is a sequence $\big\{\mathfrak{P}(1,\,n)\big\}_{n=1}^{\infty}$ of closed covers of X such that $\mathfrak{P}(1,\,n) = \mathsf{U}_{m=1}^{\infty}\,\mathfrak{P}_m(1,\,n)$ and conditions (1) through (4) of Lemma 2.2 are satisfied, with the obvious notational changes. It follows by induction that, for each $k\in Z^+$, we can find a sequence $\big\{\mathfrak{U}(k,\,n)\big\}_{n=1}^{\infty}$ of open covers and a sequence $\big\{\mathfrak{P}(k,\,n)\big\}_{n=1}^{\infty}$ of closed covers of X such that $\mathfrak{U}(k,\,n) = \big\{U_\alpha(k,\,n): \alpha\in\Gamma\big\}$, $\mathfrak{P}(k,\,n) = \mathsf{U}_{m=1}^{\infty}\,\mathfrak{P}_m(k,\,n)$, and the following is true for each $m,\,n\in Z^+$:

(1) $\mathfrak{P}_m(k,\,n) = \big\{P_{\alpha,m}(k,\,n): \alpha\in\Gamma\big\}$ is a closure-preserving collection.

(2) $P_{\alpha,m}(k,\,n)\subset U_\alpha(k,\,n)$,

(3) $P_{\alpha,m}(k,\,n)\subset P_{\alpha,m+1}(k,\,n)$,

(4) $P_{\alpha,m}(k,\,n+1)\subset P_{\alpha,m}(k,\,n)$,

(5) $U_\alpha(k+1,\,n) = U_\alpha - \mathsf{U}_{\beta<\alpha}\,P_{\beta,n}(k,\,1)$.

Notice that for $x\in X$, if $\alpha(x)$ is the first element of Γ such that

$x \in U_{\alpha(x)}$, then $x \in U_{\alpha(x)}(k, n)$ for any k, $n \in Z^+$. Hence $\mathfrak{U}(k, n)$ is indeed an open cover of X.

For k, m, $n \in Z^+$ and $\alpha \in \Gamma$ define

$$L_\alpha(k, m, n) = P_{\alpha,n}(k, 1) \cap P_{\alpha,m}(k + 1, n).$$

Then $\mathfrak{L}(k, m, n) = \{L_\alpha(k, m, n) : \alpha \in \Gamma\}$ is a closure-preserving collection of closed sets. Let γ, $\alpha \in \Gamma$ with $\gamma \neq \alpha$; say $\gamma < \alpha$. Then

$$L_\alpha(k, m, n) \subset P_{\alpha,m}(k + 1, n) \subset U_\alpha(k + 1, n)$$
$$= U_\alpha - \bigcup_{\beta < \alpha} P_{\beta,n}(k, 1) \subset U_\alpha - P_{\gamma,n}(k, 1) \subset U_\alpha - L_\gamma(k, m, n).$$

Hence $L_\gamma(k, m, n) \cap L_\alpha(k, m, n) = \varnothing$ when $\gamma \neq \alpha$. Thus $\mathfrak{L}(k, m, n)$ is a discrete collection of closed sets for each k, m, $n \in Z^+$. Clearly $L_\alpha(k, m, n) \subset U_\alpha$, so we are through if we show $\mathfrak{L} = \bigcup_{k=1}^\infty \bigcup_{m=1}^\infty \bigcup_{n=1}^\infty \mathfrak{L}(k, m, n)$ covers X.

Let $x \in X$. Since Γ is well ordered, there is a $\beta \in \Gamma$, and k, m, $n \in Z^+$, such that $x \in P_{\beta,n}(k, m)$ but $x \notin P_{\alpha,n'}(k', m')$ if $\alpha < \beta$ and k', m', $n' \in Z^+$. We show that $x \in L_\beta(k, m, n)$. First notice that, if $\alpha > \beta$, $P_{\alpha,m}(k+1, n) \subset U_\alpha(k+1, n) \subset U_\alpha - P_{\beta,n}(k, 1)$ and so $P_{\beta,n}(k, 1) \cap P_{\alpha,m}(k+1, n) = \varnothing$. Thus

$$x \in P_{\beta,n}(k, m) - \bigcup_{\alpha < \beta} P_{\alpha,m}(k + 1, n)$$
$$\subset P_{\beta,n}(k, 1) - \bigcup_{\alpha < \beta} P_{\alpha,m}(k + 1, n)$$
$$= P_{\beta,n}(k, 1) - \bigcup_{\alpha \neq \beta} P_{\alpha,m}(k + 1, n)$$
$$\subset P_{\beta,n}(k, 1) \cap P_{\beta,m}(k + 1, n) = L_\beta(k, m, n),$$

and that completes the proof.

3. **Properties of subparacompact spaces.** This section is devoted to showing a few properties of subparacompact spaces. Most of the results follow in a straightforward manner by the use of Theorem 1.2. Theorems 3.1 and 3.2 answer questions asked by Arhangel'skiĭ in [2].

THEOREM 3.1. *If $f: X \to Y$ is a closed map from a subparacompact space X onto Y then Y is subparacompact.*

PROOF. If $\mathfrak{U}$ is any open cover of Y, $f^{-1}(\mathfrak{U}) = \{f^{-1}(U) : U \in \mathfrak{U}\}$ is an open cover of X. Since X is subparacompact $f^{-1}(\mathfrak{U})$ has a closed refinement $\mathcal{P} = \bigcup_{n=1}^\infty \mathcal{P}_n$ where $\mathcal{P}_n$ is a closure-preserving collection of closed sets for each $n \in Z^+$. Since f is a closed map it follows that $f(\mathcal{P}_n) = \{f(P) : P \in \mathcal{P}_n\}$ is a closure-preserving collection of closed sets

in Y. Thus $f(\mathcal{P})$ is a σ-closure-preserving refinement of $\mathcal{U}$ and Y is subparacompact.

THEOREM 3.2. *If $f: X \to Y$ is a perfect map from a regular space X onto a subparacompact space Y then X is subparacompact.*

PROOF. Suppose $\mathcal{U}$ is any open cover of X. For each $y \in Y$ we can find a finite subcollection $\mathcal{U}(y) \subset \mathcal{U}$ such that $f^{-1}(y) \subset U(y) = \bigcup \{ U : U \in \mathcal{U}(y) \}$. Let $G(y) = Y - f(X - U(y))$; then $\mathcal{G} = \{ G(y) : y \in Y \}$ is an open cover of Y. Consequently $\mathcal{G}$ has a σ-discrete refinement $\mathcal{P} = \bigcup_{n=1}^{\infty} \mathcal{P}_n$ where each $\mathcal{P}_n$ is a discrete collection. It follows that $f^{-1}(\mathcal{P}) = \{ f^{-1}(P) : P \in \mathcal{P} \}$ is a σ-discrete refinement of $\{ U(y) : y \in Y \}$. Given $P \in \mathcal{P}$ let $y(P)$ be a fixed element of Y such that $f^{-1}(P) \subset U(y(P))$. For $n \in Z^+$ let $\mathcal{B}_n = \{ f^{-1}(P) \cap U : P \in \mathcal{P}_n, U \in \mathcal{U}(y(P)) \}$. Since $f^{-1}(\mathcal{P}_n)$ is a discrete collection in X each $x \in X$ has a neighborhood N_x which intersects at most one element of $f^{-1}(\mathcal{P}_n)$. Since each element of $f^{-1}(\mathcal{P}_n)$ intersects only finitely many elements of $\mathcal{B}_n$ and each element of $\mathcal{B}_n$ is contained in some element of $\mathcal{P}_n$ it follows that N_x will intersect only finitely many elements of $\mathcal{B}_n$. So $\mathcal{B}_n$ is locally-finite and $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ is a σ-locally-finite refinement of $\mathcal{U}$. Since X is regular it follows that every open cover of X has a σ-locally-finite closed refinement. Hence X is subparacompact.

A completely regular space X is called a *p-space* [1] if in the Stone-Čech compactification $\beta(X)$ there is a sequence $\{ \gamma_n \}_{n=1}^{\infty}$ of open covers of X such that $\bigcap_{n=1}^{\infty} \operatorname{St}(x, \gamma_n) \subset X$ for each $x \in X$. The sequence $\{ \gamma_n \}_{n=1}^{\infty}$ is called a *pluming* for X in $\beta(X)$. X is a *strict p-space* if there is a pluming $\{ \gamma_n \}_{n=1}^{\infty}$ with the additional property: $\bigcap_{n=1}^{\infty} \operatorname{Cl}(\operatorname{St}(x, \gamma_n)) = \bigcap_{n=1}^{\infty} \operatorname{St}(x, \gamma_n)$ for each $x \in X$.

By Theorem 3.2 a regular space which is the perfect preimage of a developable space is subparacompact; if this preimage is completely regular it can be shown to be a p-space. Since Arhangel'skiĭ [2] announced that any σ-paracompact p-space can be mapped perfectly onto a developable space, we have the following corollary to Theorem 3.2.

COROLLARY 3.3. *A completely regular space X can be mapped onto a developable space by a perfect map if and only if X is a subparacompact p-space.*

In [2] Arhangel'skiĭ asks whether every strict p-space can be mapped perfectly onto a developable space. To answer this question negatively it suffices to find an example of a strict p-space which is not subparacompact. Such an example will be given in the last section of this paper.

THEOREM 3.4. *A collectionwise normal subparacompact space is paracompact.*

THEOREM 3.4 is a restatement of a result by McAuley [9] that a collectionwise normal F_σ-screenable space is paracompact.

Since σ-discrete families in a countably compact space must be countable the following is obvious:

THEOREM 3.5. *A countably compact, subparacompact space X is compact.*

THEOREM 3.6. *A subparacompact locally-developable space X is developable.*

PROOF. Suppose each point $x \in X$ has an open neighborhood N_x with a development $\{\mathcal{G}_n(x)\}_{n=1}^{\infty}$. Let $\mathcal{P} = \bigcup_{n=1}^{\infty} \mathcal{P}_n$ be a σ-discrete closed refinement of $\{N_x : x \in X\}$ where each $\mathcal{P}_n$ is a discrete collection.

Now fix an arbitrary positive integer n. Given $P \in \mathcal{P}_n$ let $x(P)$ be a fixed element of X such that $P \subset N_{x(P)}$ and let

$$U(P) = X - \bigcup \{P' \in \mathcal{P}_n : P' \neq P\}.$$

For $m \in Z^+$ let

$$\mathcal{U}_{n,m}(P) = \{U(P) \cap G : G \in \mathcal{G}_m(x(P))\}.$$

Finally we define

$$\mathcal{U}_{n,m} = \{U : U \in \mathcal{U}_{n,m}(P), P \in \mathcal{P}_n\} \cup \{Q_n\}$$

where $Q_n = X - \bigcup \{P : P \in \mathcal{P}_n\}$. Then $\mathcal{U}_{n,m}$ is an open cover of X for each $n, m \in Z^+$ and we show that $\{\mathcal{U}_{n,m} : n, m \in Z^+\}$ is a development for X.

If $z \in X$ there is an integer $n \in Z^+$ with some $P \in \mathcal{P}_n$ such that $z \in P$. Consequently if O is any open set containing z there is some $m \in Z^+$ such that $\mathrm{St}(z, \mathcal{G}_m(x(P))) \subset O \cap N_{x(P)}$. By construction z is not contained in any element of $\mathcal{U}_{n,m}(P')$ for any $P' \in \mathcal{P}_n$ such that $P' \neq P$. Thus $\mathrm{St}(z, \mathcal{U}_{n,m}) = \mathrm{St}(z, \mathcal{U}_{n,m}(P)) \subset \mathrm{St}(z, \mathcal{G}_m(x(P))) \subset O \cap N_{x(P)} \subset O$. Hence $\mathrm{St}(z, \mathcal{U}_{n,m}) \subset O$ and the theorem is proved.

The proof of the following theorem is straightforward and is left to the reader.

THEOREM 3.7. *If X is a countable union of closed subparacompact subspaces then X is subparacompact.*

4. Examples.

EXAMPLE 4.1. A strict p-space which is not subparacompact and hence cannot be mapped perfectly onto a developable space.

Let ω_1 (resp. ω_2) be the first ordinal of cardinality $\aleph_1$ (resp. $\aleph_2$). If X is the set of points in the cartesian product of $[0, \omega_2)$ with itself and $\alpha \in (0, \omega_2)$ we define:

$$L_{1,\alpha} = \{(\alpha, \beta) \in X: \beta \in [0, \omega_2)\} \text{ and } L_{2,\alpha} = \{(\beta, \alpha) \in X: \beta \in [0, \omega_2)\}.$$

For convenience let $L_{1,0} = L_{2,0} = \{(0, 0)\}$.

Define an open base for a topology on X as follows: For each $(\alpha, \beta) \in X$ where $\alpha \neq 0$ and $\beta \neq 0$ the singleton set $\{(\alpha, \beta)\}$ is in the base. For $i = 1, 2$ and $\alpha \in [0, \omega_2)$ the base includes all subsets of $L_{i,\alpha}$ which have a finite complement relative to $L_{i,\alpha}$. It is easily verified that this collection does form an open base for a topology on X.

Notice that with the above described topology X is a T_2 locally compact space. Hence X is open in $\beta(X)$. That X is a strict p-space follows from the definition if, for each $n \in Z^+$, we let

$$\mathcal{G} = \gamma_n = \{L_{i,\alpha}: \alpha \in [0, \omega_2), \quad i = 1, 2\}.$$

Then γ_n is a cover of X open in $\beta(X)$ and $\mathrm{St}(x, \gamma_n)$ is an open compact set for each $x \in X$.

The following lemma will be needed to show that X is not subparacompact.

LEMMA. *Suppose A and B are subsets of X with the property: $A \cap L_{1,\alpha}$ and $B \cap L_{2,\alpha}$ are countable for each $\alpha \in [0, \omega_2)$. Then A and B do not cover X.*

PROOF. Suppose $X = A \cup B$. Let

$$\beta_0 = \sup\{\beta: (\alpha, \beta) \in A \cap L_{1,\alpha}, \alpha \in [0, \omega_1)\}.$$

Then $\beta_0 < \omega_2$ and it follows that $[0, \omega_1) \times (\beta_0, \omega_2)$ is a nonempty subset of X disjoint from A. Thus $[0, \omega_1) \times (\beta_0, \omega_2) \subset B$. Let $\alpha_0 \in (\beta_0, \omega_2)$. Then $[0, \omega_1) \times \{\alpha_0\} \subset B \cap L_{2,\alpha_0}$ which is in contradiction with the original condition placed on B.

We show that X is not subparacompact by showing that $\mathcal{G}$ cannot have a σ-discrete refinement.

Assume $\mathcal{P} = \bigcup_{n=1}^{\infty} \mathcal{P}_n$ is a refinement of $\mathcal{G}$ where each $\mathcal{P}_n$ is a discrete collection. Let $\mathcal{G}' = \{L_{1,\alpha}: \alpha \in [0, \omega_2)\}$ and $\mathcal{G}'' = \{L_{2,\alpha}: \alpha \in [0, \omega_2)\}$; then $\mathcal{G}'$ and $\mathcal{G}''$ are subcollections of $\mathcal{G}$. Let $\mathcal{P}_n'$ and $\mathcal{P}_n''$ be all elements of $\mathcal{P}_n$ which are contained in elements of $\mathcal{G}'$ and $\mathcal{G}''$ respectively. Note that every element of $\mathcal{P}_n$ will be in $\mathcal{P}_n'$ or $\mathcal{P}_n''$. Let P_n' and P_n'' be the union of all the elements in $\mathcal{P}_n'$ and $\mathcal{P}_n''$ respectively. If $\alpha \in [0, \omega_2)$ there is a neighborhood of $(\alpha, 0)$ which intersects at most one element of $\mathcal{P}_n''$. Hence $L_{1,\alpha}$ intersects only finitely many elements of $\mathcal{P}_n''$ and it follows that $P_n'' \cap L_{1,\alpha}$ is finite. Similarly $P_n' \cap L_{2,\alpha}$ is finite. Let $A = \bigcup_{n=1}^{\infty} P_n''$ and $B = \bigcup_{n=1}^{\infty} P_n'$. Then A and B satisfy the conditions

in the lemma and consequently cannot cover X. Thus $\mathcal{P}$ does not cover X.

EXAMPLE 4.2. A pointwise paracompact space which is not subparacompact.

It is easily verified that the space of Example 4.1 is pointwise paracompact.

EXAMPLE 4.3. A pseudocompact subparacompact space which is not countably compact.

Let X be the set of points in the unit interval $[0, 1]$ with nonzero points having their usual neighborhoods. Let the neighborhood system at zero include the usual neighborhoods minus the points in the sequence $\{1/n\}_{n=1}^{\infty}$.

EXAMPLE 4.4. A normal, countably paracompact, subparacompact space which is not paracompact.

Let X be the space of Example H described by Bing in [3]. This space is perfectly normal and hence countably paracompact [7]. X is subparacompact since it can be shown to possess a σ-discrete network; however it is not paracompact since it is not collectionwise normal.

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