A FIXED POINT THEOREM FOR MAPPINGS WITH A CONTRACTIVE ITERATE

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1. Let \((X, d)\) be a metric space and \(f: X \to X\) a mapping. If \(f\) satisfies the condition

\[
(1) \quad d(f(x), f(y)) \leq kd(x, y)
\]

for all \(x, y \in X\) and for some \(k < 1\), then \(f\) is called a contraction.

A well-known theorem of Banach [1] states that for \(X\) complete, each contraction mapping \(f\) has a unique fixed point \(u\), and the successive approximations \(\{f^n(x_0)\}\) converge to \(u\) for each \(x_0 \in X\). In this paper, we investigate mappings which are not necessarily contractions and prove the following result.

**Theorem.** Let \((X, d)\) be a complete metric space, and \(f: X \to X\) a continuous mapping satisfying the condition: there exists a \(k < 1\) such that for each \(x \in X\), there is a positive integer \(n(x)\) such that for all \(y \in X\)

\[
(2) \quad d(f^n(x)(y), f^n(x)(x)) \leq kd(y, x).
\]

Then \(f\) has a unique fixed point \(u\) and \(f^n(x_0) \to u\) for each \(x_0 \in X\).

**Lemma.** If \(f: X \to X\) be any mapping satisfying the condition of the above theorem then for each \(x \in X\), \(r(x) = \sup_n d(f^n(x), x)\) is finite.

**Proof.** Let \(x \in X\) and let

\[
l(x) = \max\{d(f^k(x), x) : k = 1, 2, \ldots, n(x)\}.
\]

If \(n\) is a positive integer, there exists an integer \(s \geq 0\) such that \(s \cdot n(x) < n \leq (s + 1) \cdot n(x)\), and

\[
d(f^n(x), x) \leq d(f^n(x), f^{n-n(x)}(x), f^n(x), x) + d(f^n(x), x)
\]

\[
\leq kd(f^{n-n(x)}(x), x) + l(x)
\]

\[
\leq l(x) + kl(x) + k^2l(x) + \cdots + k^sl(x)
\]

\[
\leq l(x)/(1 - k) \quad \text{for all } n \geq 0.
\]

Hence \(r(x) = \sup_n d(f^n(x), x)\) is finite.

**Proof of the Theorem.** Let \(x_0 \in X\) be arbitrary. Let \(m_0 = n(x_0), x_1 = f^{m_0}(x_0)\) and inductively \(m_i = n(x_i), x_{i+1} = f^{m_i}(x_i)\). We show that

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the sequence \( \{x_n\} \) is a convergent sequence. By routine calculation we have
\[
d(x_{n+1}, x_n) = d(f^{m-1}f(x_{n-1}), f^{m-1}(x_{n-1})) \leq k d(f^{m}(x_{n-1}), x_{n-1}) \leq \cdots \\
\leq k^n d(f^{m}(x_0), x_0).
\]
Therefore, it follows by lemma that \( d(x_{n+1}, x_n) \leq k^n r(x_0) \). Thus, for \( m > n \),
\[
d(x_m, x_n) \leq \sum_{i=n}^{m-1} d(x_{i+1}, x_i) \leq \frac{k^n}{1-k} r(x_0) \to 0 \quad \text{as} \quad n \to \infty.
\]
The sequence \( \{x_n\} \) is therefore Cauchy. Let \( x_n \to u \in X \).

If \( f(u) \neq u \), then, there exists a pair of disjoint closed neighborhoods \( U \) and \( V \) such that \( u \in U, f(u) \in V \) and
\[
(3) \quad \rho = \text{Inf}\{d(x, y) : x \in U, y \in V\} > 0.
\]
Since \( f \) is continuous, \( x_n \in U \) and \( f(x_n) \in V \) for all \( n \) sufficiently large. However,
\[
d(f(x_n), x_n) = d(f^{m-1}f(x_{n-1}), f^{m-1}(x_{n-1})) \leq k d(f^{m}(x_{n-1}), x_{n-1}) \leq \cdots \\
< k^n d(f(x_0), x_0) \to 0
\]
as \( n \to \infty \), contradicting (3). Thus \( f(u) = u \).

The uniqueness of the fixed point follows immediately from (2). To show that \( f^n(x_0) \to u \), set
\[
\rho_* = \text{max}\{d(f^m(x_0), u) : m = 0, 1, 2 \cdots, (n(u) - 1)\}.
\]
If \( n \) is a sufficiently large integer, then \( n = r \cdot n(u) + q, 0 \leq q < n(u), r > 0 \), and
\[
d(f^n(x_0), u) = d(f^{r \cdot n(u) + q}(x_0), f^{n(u)}(u)) \leq k d(f^{(r-1) \cdot n(u) + q}(x_0), u) \leq \cdots \\
\leq k^r d(f^q(x_0), u) \leq k^r \rho_*.
\]
Since \( n \to \infty \) implies \( r \to \infty \), we have \( d(f^n(x_0), u) \to 0 \) as \( n \to \infty \). This establishes the theorem.

2. In this section we give an example of a continuous function \( f \) which satisfies condition (2), but is not a contraction. The author is thankful to the referee for showing that no iterate of \( f \) is a contraction. Bryant [2] considers a function \( f \) (not necessarily continuous) on a complete metric space \( X \) into itself for which there exists a \( k, 0 \leq k < 1 \), and a positive integer \( n \), such that \( d(f^n(x), f^n(y)) \leq k d(x, y) \) for all \( x, y \in X \). That Bryant’s condition is stronger than condition (2) is shown below. The author is grateful to Professor David R. Anderson for his help in the following example.
Example. Let $X$ be the closed unit interval $[0, 1]$ with the usual metric. Write

$$X = \bigcup_{n=1}^{\infty} \left[ \frac{1}{2^n}, \frac{1}{2^{n-1}} \right] \cup \{0\},$$

and let $f: X \to X$ be defined as follows.

For each $n = 1, 2 \cdots$, let

$$f: \left[ \frac{1}{2^n}, \frac{1}{2^{n-1}} \right] \to \left[ \frac{1}{2^{n+1}}, \frac{1}{2^n} \right]$$

be defined by

$$f(x) = \begin{cases} 
\frac{n+2}{n+3} \left(x - \frac{1}{2^{n-1}}\right) + \frac{1}{2^n}, & \text{if } x \in \left[ \frac{3n+5}{2^{n+1}(n+2)}, \frac{1}{2^{n-1}} \right] \\
\frac{1}{2^{n+1}}, & \text{if } x \in \left[ \frac{1}{2^n}, \frac{3n+5}{2^{n+1}(n+2)} \right]
\end{cases}$$

and let $f(0) = 0$.

It is obvious that $f$ is a nondecreasing, continuous function on $[0, 1]$ with $0$ as the only fixed point, and that $f$ is not a contraction. If $x \in [1/2^n, 1/2^{n-1}]$ and $y \in X$, then by a routine examination of cases $y \in [1/2^m, 1/2^{m-1}]$ for $m \geq n$ and $m \leq n$, it is easy to verify that $f$ satisfies

$$|f(x) - f(y)| \leq \frac{n+3}{n+4} |x-y| \quad \text{for all } y \in X.$$

Therefore, if we choose $k = 1/2$ in (2), then for each $x \in [1/2^n, 1/2^{n-1}]$, $n(x)$ may be taken as $n+3$, whereas $n(0)$ may be taken as any integer greater than one.

To show that the condition in [2] is stronger than (2), let $0 \leq k < 1$, and $N$ (a natural number) be given; it will be shown that there exists $x$ and $y$ such that $|f^N(x) - f^N(y)| > k|x-y|$. Choose and fix $n > (Nk/(1-k)) - 2$. Since $f^i$ is uniformly continuous on $[0, 1]$ for $i = 1, 2, \cdots, N$, there is some $\delta > 0$ such that

$$|x - y| < \delta \Rightarrow |f^i(x) - f^i(y)| < \frac{n + N + 3}{(n + N + 2)2^{n+N+1}}$$

for $i = 1, 2, \cdots, N$. Setting $x = 1/2^{n-1}$ and $y$ any member of $[1/2^n, 1/2^{n-1}]$ such that $0 < |x-y| < \delta$, it can be shown that $f^i(x)$ and $f^i(y)$ are both members of
\[
\begin{bmatrix}
\frac{3(n + i) + 5}{2^{n+i+1}(n + i + 2)} & 1 \\
\frac{1}{2^{n+i-1}} & 1
\end{bmatrix}
\]

for \(i = 1, 2, \cdots, N\). Thus

\[
|f(x) - f(y)| = \frac{n + 2}{n + 3} |x - y|,
\]

\[
|f^2(x) - f^2(y)| = \frac{n + 2}{n + 4} |x - y|, \cdots, |f^N(x) - f^N(y)|
\]

\[
= \frac{n + 2}{n + 2 + N} |x - y| > k |x - y|.
\]

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**References**


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