Let \( \psi(z) \) be a univalent meromorphic function defined on \( U \), the unit disc, and let \( S^2 \) denote the Riemann sphere, that is, the one point compactification of the complex plane \( \mathbb{R}^2 \). Let \( C(\psi, e^{it}) \) be the set of all points \( \xi \) in \( S^2 \) such that there exists a sequence \( z_n \to e^{it} \) with \( \psi(z_n) \) clustering at \( \xi \). It is easily seen that \( C(\psi, e^{it}) \subseteq \partial \psi(U) \), and Furthermore, \( \partial \psi(U) \) has at least two points. If \( B(z) \) is a finite Blaschke product, and \( f(z) = \psi(B(z)) \), then \( C(f, e^{it}) \subseteq \partial f(U) \). If \( B(z) \) is an infinite Blaschke product, this property is destroyed since \( \psi(0) \in C(f, e^{it}) \), where the zeros of \( B(z) \) cluster at \( e^{it} \).

The purpose of this paper is to present the following theorem, characterizing those meromorphic functions “clustering on the boundary”:

**Theorem.** Let \( f(z) \) be meromorphic in \( U \). Then \( C(f, e^{it}) \subseteq \partial f(U) \), for all \( e^{it} \), with \( \partial f(U) \) consisting of at least two points, if and only if \( f(z) = \psi(B(z)) \), where \( \psi(z) \) is a univalent meromorphic function and \( B(z) \) a finite Blaschke product.

**Proof.** We first show \( f(U) \) is simply connected, that is, \( K = S^2 - f(U) \) is connected. We may assume \( K \) is compact in \( \mathbb{R}^2 \). (Let \( z_0 \in f(U), T(z) = 1/(z - z_0) \), and \( g(z) = T(f(z)) \). Then \( g(U) \) contains \( \infty \) and \( C(g, e^{it}) \subseteq \partial g(U) \). If \( g(z) = \psi(B(z)) \), then \( f(z) = T^{-1} \psi(B(z)) \).) Suppose that \( K \) has two different components \( K_1 \) and \( K_2 \), and let \( \xi_i \in \partial f(U) \cap K_i, i = 1, 2 \). By the Zoretti Theorem [3, p. 35], there exists a Jordan curve \( J \) such that \( J \cap K = \emptyset \) and \( J \) separates \( K_1 \) and \( K_2 \). Since \( \xi_i \in \partial f(U) \), there exists \( e^{it_i} \) such that \( \xi_i \in C(f, e^{it_i}) \), and since this latter set is connected, it is contained in \( K_i \). Choose sequences \( z_n \to e^{it_i} \) and \( z_n' \to e^{it_i} \) such that

1. \( |z_n| / \gamma_1 \) and \( |z_n'| / \gamma_1 \),
2. \( f(z_n) \in \text{interior of } J \) and \( f(z_n') \in \text{exterior of } J \).

Let \( \gamma_n \) be the circular arc \( z(t) = |z_n| e^{it}, t \) varying between \( \arg z_n \) and \( \arg z_n' \), together with the segment \([z_n', |z_n| e^{i \arg z_n'}]\). By (2), we may choose \( w_n \in \gamma_n \) such that \( f(w_n) \in J \). If \( w_n \) clusters at \( e^{it}, f(w_n) \) clusters in \( J \), a subset of \( f(U) \), and therefore, \( C(f, e^{it}) \cap f(U) \neq \emptyset \), a contradiction. Hence \( K \) is connected, and consequently \( f(U) \) is simply connected.

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Let \( \psi \) be a 1-1 meromorphic function from \( U \) onto \( f(U) \) guaranteed by the Riemann mapping theorem, and consider \( B(z) = \psi^{-1}(f(z)) \). If \( z_n \in U \) and \( z_n \to e^{it} \) with \( B(z_n) \to w \) and \( |w| < 1 \), then \( \psi(w) = \lim_{n \to \infty} f(z_n) \in f(U) \), contradicting the hypothesis that \( C(f, e^{it}) \subset \partial f(U) \). Hence
\[
\min \left| B(re^{it}) \right| \to 1 \quad \text{as} \quad r \to 1.
\]
Therefore, \( B(z) \) has finitely many zeros, and \( C(B, e^{it}) \subset \{ z : |z| = 1 \} \), for all \( e^{it} \). It is now easy to show that \( B(z) \) is a unimodular constant times the finite Blaschke product formed by the zeros of \( B(z) \), \( B_1(z) \), by applying the maximum modulus principle to \( B/B_1 \) and \( B_1/B \).

**Corollary 1.** If \( f(z) \) is meromorphic in \( U \) with \( C(f, e^{it}) \subset \partial f(U) \) for all \( e^{it} \), then,

1. \( f(z) \) is \( m \)-valent for some positive integer \( m \);
2. \( f \in H^p \), \( 0 < p < 1/2 \); and
3. \( f(z) \) converges nontangentially as \( z \to e^{it} \) except possibly for a set of capacity zero.

**Proof.** (1) is true since \( B(z) \) is \( m \)-valent.
(2) and (3) follow since it is well known that a univalent function possesses properties (2) and (3). (See [1, p. 473], and [2, p. 56].)

**Corollary 2.** Let \( f(z) \) be meromorphic in \( U \) with \( f(U) \) multiply connected. Then there exists a point \( e^{it} \) such that \( C(f, e^{it}) \cap f(U) \neq \emptyset \).

**Bibliography**


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