MEROMORPHIC FUNCTIONS WHICH CLUSTER ON THE BOUNDARY

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Let $\psi(z)$ be a univalent meromorphic function defined on U, the unit disc, and let S^2 denote the Riemann sphere, that is, the one point compactification of the complex plane $\boldsymbol{R}^2$. Let $C(\psi, e^{it})$ be the set of all points ζ in S^2 such that there exists a sequence $z_n \to e^{it}$ with $\psi(z_n)$ clustering at ζ. It is easily seen that $C(\psi, e^{it}) \subset \partial\psi(U)$, and furthermore, $\partial\psi(U)$ has at least two points. If $B(z)$ is a finite Blaschke product, and $f(z) = \psi(B(z))$, then $C(f, e^{it}) \subset \partial f(U)$. If $B(z)$ is an infinite Blaschke product, this property is destroyed since $\psi(0) \in C(f, e^{it})$, where the zeros of $B(z)$ cluster at e^{it}.

The purpose of this paper is to present the following theorem, characterizing those meromorphic functions "clustering on the boundary":

THEOREM. *Let $f(z)$ be meromorphic in U. Then $C(f, e^{it}) \subset \partial f(U)$, for all e^{it}, with $\partial f(U)$ consisting of at least two points, if and only if $f(z) = \psi(B(z))$, where $\psi(z)$ is a univalent meromorphic function and $B(z)$ a finite Blaschke product.*

PROOF. We first show $f(U)$ is simply connected, that is, $K = S^2 - f(U)$ is connected. We may assume K is compact in $\boldsymbol{R}^2$. (Let $z_0 \in f(U)$, $T(z) = 1/(z - z_0)$, and $g(z) = T(f(z))$. Then $g(U)$ contains ∞ and $C(g, e^{it}) \subset \partial g(U)$. If $g(z) = \psi(B(z))$, then $f(z) = T^{-1} \circ \psi(B(z))$.) Suppose that K has two different components K_1 and K_2, and let $\zeta_i \in \partial f(U) \cap K_i$, $i = 1, 2$. By the Zoretti Theorem [3, p. 35], there exists a Jordan curve J such that $J \cap K = \varnothing$ and J separates K_1 and K_2. Since $\zeta_i \in \partial f(U)$, there exists e^{it_i} such that $\zeta_i \in C(f, e^{it_i})$, and since this latter set is connected, it is contained in K_i. Choose sequences $z_n \to e^{it_1}$ and $z_n' \to e^{it_2}$ such that

(1) $|z_n| \nearrow 1$ and $|z_n'| \nearrow 1$,

(2) $f(z_n) \in$ interior of J and $f(z_n') \in$ exterior of J.

Let γ_n be the circular arc $z(t) = |z_n| e^{it}$, t varying between $\arg z_n$ and $\arg z_n'$, together with the segment $[z_n', |z_n| e^{i \arg z_n'}]$. By (2), we may choose $w_n \in \gamma_n$ such that $f(w_n) \in J$. If w_n clusters at e^{it}, $f(w_n)$ clusters in J, a subset of $f(U)$, and therefore, $C(f, e^{it}) \cap f(U) \neq \varnothing$, a contradiction. Hence K is connected, and consequently $f(U)$ is simply connected.

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Let ψ be a 1-1 meromorphic function from U onto $f(U)$ guaranteed by the Riemann mapping theorem, and consider $B(z)=\psi^{-1}(f(z))$. If $z_n\in U$ and $z_n\to e^{it}$ with $B(z_n)\to w$ and $|w|<1$, then $\psi(w)=\lim_{n\to\infty} f(z_n)\in f(U)$, contradicting the hypothesis that $C(f, e^{it})\subset\partial f(U)$. Hence

$$\min_{\theta} |B(re^{i\theta})| \to 1 \quad \text{as } r\to 1.$$

Therefore, $B(z)$ has finitely many zeros, and $C(B, e^{it})\subset\{z: |z|=1\}$, for all e^{it}. It is now easy to show that $B(z)$ is a unimodular constant times the finite Blaschke product formed by the zeros of $B(z)$, $B_1(z)$, by applying the maximum modulus principle to B/B_1 and B_1/B.

COROLLARY 1. *If $f(z)$ is meromorphic in U with $C(f, e^{it})\subset\partial f(U)$ for all e^{it}, then,*

(1) *$f(z)$ is m-valent for some positive integer m;*

(2) *$f\in H^p$, $0<p<1/2$; and*

(3) *$f(z)$ converges nontangentially as $z\to e^{it}$ except possibly for a set of capacity zero.*

PROOF. (1) is true since $B(z)$ is m-valent.

(2) and (3) follow since it is well known that a univalent function possesses properties (2) and (3). (See [1, p. 473], and [2, p. 56].)

COROLLARY 2. *Let $f(z)$ be meromorphic in U with $f(U)$ multiply connected. Then there exists a point e^{it} such that $C(f, e^{it})\cap f(U)\neq\emptyset$.*

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