THE REPRESENTATION OF CHAINABLE CONTINUA
WITH ONLY TWO BONDING MAPS

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Definition. A set $S$ of continuous functions of $[0, 1]$ into $[0, 1]$ is called a "complete" set of bonding maps if every chainable continuum can be obtained as the inverse limit of an inverse mapping system each of whose coordinate spaces is $[0, 1]$ and each of whose bonding maps is in $S$.

It is shown in [1] that a dense set is complete. Jolly and Rogers in [3] demonstrate a complete set with only four elements. It follows from the theorem of Jarník and Knichal [2] that there is a complete set with only two elements. Mahavier proves in [4] that every complete set must have at least two elements and therefore two is the minimum number.

In this note, we prove the following:

Theorem. There exists a continuous function $f$ of $[0, 1]$ into $[0, 1]$ such that $\{f, \frac{1}{2}f\}$ is complete.

First we will establish a lemma, the proof of which closely follows [2]. Let $C$ denote the set of all continuous functions of $[0, 1]$ into $[0, 1]$. $e(x) = \frac{1}{2}x$, $e^{n+1}(x) = e(e^n(x))$ and $e^0(x) = x$.

Lemma. If $B_1, B_2 \in C$, then there exists $g \in C$ such that $g e g e^3 = B_1$ and $g e^2 g e^3 = B_2$.

Proof. Let $g$ be defined as follows: $g(x) = 3 \cdot 2^{-2} + 2x$ for $0 \leq x \leq 2^{-3}$, $g(x) = \theta_2(16x - 3)$ for $3 \cdot 2^{-4} \leq x \leq 2^{-2}$, $g(x) = \theta_1(8x - 3)$ for $3 \cdot 2^{-3} \leq x \leq 2^{-1}$, and $g$ is linear in each of the intervals $[2^{-3}, 3 \cdot 2^{-4}]$, $[2^{-2}, 3 \cdot 2^{-3}]$, and $[2^{-1}, 1]$. It is now easy to verify that $g e g e^3 = \theta_1$ and $g e^2 g e^3 = \theta_2$.

Proof of the theorem. It follows from the lemma and a remark made previously that there exists $g \in C$ such that $\{g e g e^3, g e^2 g e^3\}$ is complete. Thus if $M$ is a chainable continuum, then $M$ can be represented in the form $M \cong g e^{n_1} g e^{n_2} g e^{n_3} g e^{n_4} g e^{n_4} e^{n_5} \cdots$. The diagram arrows are omitted to save space. Each of the exponents $n_i$ is 1 or 2. Let $f = e g e^3 g$ and then by regrouping the bonding maps and deleting the first map, we obtain $M \cong e^{n_i-1} e^{n_i-1} e^{n_i-1} e^{n_i-1} f \cdots$. Now for each $i$, $n_i - 1 = 0$ or 1 and so another regrouping yields $M \cong (a_1 f)(a_2 f) \cdots$ where for each $i$, $a_i = 1$ or $\frac{1}{2}$. This completes the proof.
It should be noted that \( \{g e g^3, g e^2 g^3\} \) is complete as a consequence of the fact that the collection of all finite compositions of these functions is dense in \( C \). The collection of all finite compositions of the functions \( f \) and \( \frac{1}{2} f \) is not dense in \( C \) since the range of \( f \) is a proper subinterval of \( [0, 1] \).

References


3. R. F. Jolly and J. T. Rogers, Jr., Inverse limit spaces defined by only finitely many distinct bonding maps, Fund. Math. (to appear)