THE REPRESENTATION OF CHAINABLE CONTINUA
WITH ONLY TWO BONDING MAPS

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Definition. A set \( S \) of continuous functions of \([0, 1]\) into \([0, 1]\) is called a "complete" set of bonding maps if every chainable continuum can be obtained as the inverse limit of an inverse mapping system each of whose coordinate spaces is \([0, 1]\) and each of whose bonding maps is in \( S \).

It is shown in [1] that a dense set is complete. Jolly and Rogers in [3] demonstrate a complete set with only four elements. It follows from the theorem of Jarník and Knichal [2] that there is a complete set with only two elements. Mahavier proves in [4] that every complete set must have at least two elements and therefore two is the minimum number.

In this note, we prove the following:

Theorem. There exists a continuous function \( f \) of \([0, 1]\) into \([0, 1]\) such that \( \{f, \frac{1}{2}f\} \) is complete.

First we will establish a lemma, the proof of which closely follows [2]. Let \( C \) denote the set of all continuous functions of \([0, 1]\) into \([0, 1]\). \( e(x) = \frac{1}{2}x, e_{n+1}(x) = e(e_n(x)) \) and \( e^0(x) = x \).

Lemma. If \( B_1, B_2 \in C \), then there exists \( g \in C \) such that \( g e g e^3 = B_1 \) and \( g e^2 g e^3 = B_2 \).

Proof. Let \( g \) be defined as follows: \( g(x) = 3 \cdot 2^{-2} + 2x \) for \( 0 \leq x \leq 2^{-3} \), \( g(x) = \theta_2(16x - 3) \) for \( 3 \cdot 2^{-4} \leq x \leq 2^{-2} \), \( g(x) = \theta_1(8x - 3) \) for \( 3 \cdot 2^{-3} \leq x \leq 2^{-1} \), and \( g \) is linear in each of the intervals \([2^{-3}, 3 \cdot 2^{-4}], [2^{-2}, 3 \cdot 2^{-3}] \), and \([2^{-1}, 1] \). It is now easy to verify that \( g e g e^3 = \theta_1 \) and \( g e^2 g e^3 = \theta_2 \).

Proof of the theorem. It follows from the lemma and a remark made previously that there exists \( g \in C \) such that \( \{g e g e^3, g e^2 g e^3\} \) is complete. Thus if \( M \) is a chainable continuum, then \( M \) can be represented in the form \( M \cong g^n e^3 g e^n_{q+1} g e^n_{q+2} g e^n_{q+3} \cdots \). The diagram arrows are omitted to save space. Each of the exponents \( n_i \) is 1 or 2. Let \( f = e g e^3 g \) and then by regrouping the bonding maps and deleting the first map, we obtain \( M \cong e^{-1} f e_n e^{-1} f e_n e^{-1} f \cdots \). Now for each \( i, n_i - 1 = 0 \) or 1 and so another regrouping yields \( M \cong (a_if)(a_{2f}) \cdots \) where for each \( i, a_i = 1 \) or \( \frac{1}{2} \). This completes the proof.

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It should be noted that $\{gege^3, ge^2ge^3\}$ is complete as a consequence of the fact that the collection of all finite compositions of these functions is dense in $C$. The collection of all finite compositions of the functions $f$ and $\frac{1}{2}f$ is not dense in $C$ since the range of $f$ is a proper subinterval of $[0, 1]$.

References


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