NUCLEAR TOPOLOGIES CONSISTENT WITH A DUALITY

MICHAEL B. DOLLINGER

Let E be a (Hausdorff) locally convex space, and let E' be its dual space. A subset $B \subset E'$ is said to be prenuclear if there exist a $\sigma(E', E)$-closed equicontinuous subset $A \subset E'$ and a positive Radon measure μ on A such that for each $x \in E$,

$$\sup_{y' \in B} |\langle x, y' \rangle| \leqq \int_A |\langle x, x' \rangle| \, d\mu(x').$$

Pietsch has shown that E is a nuclear space if and only if every equicontinuous subset of E' is prenuclear (see [1] or [3]). We shall use this result to characterize all nuclear topologies on a locally convex space which are consistent with a given duality. We refer the reader to [3] for the basic results and notation that we shall use.

We begin with a definition.

Definition. Let E be a locally convex space, and let $B_0 \subset E'$. We shall say that B_0 is a hypernuclear set if there exists a sequence $\{(B_n, \mu_n): n = 1, 2, \cdots\}$ where each B_n is a $\sigma(E', E)$-closed equicontinuous subset of E' and μ_n is a positive $\sigma(E', E)$-Radon measure on B_n such that for each $n \geqq 1$ and each $x \in E$,

$$\sup_{y' \in B_{n-1}} |\langle x, y' \rangle| \leqq \int_{B_n} |\langle x, x' \rangle| \, d\mu_n(x').$$

We shall say that the sequence $\{(B_n, \mu_n): n = 1, 2, \cdots\}$ corresponds to the hypernuclear set B_0.

In the following proposition, we state some elementary properties of hypernuclear sets. The proof is fairly direct, and so it has been omitted.

Proposition 1. (a) *Finite subsets of E' are hypernuclear.*

(b) *If A and B are hypernuclear and λ is a scalar, then $A \cup \lambda B$ is hypernuclear.*

(c) *If A is hypernuclear and $B \subset A$, then B is hypernuclear.*

(d) *If A is hypernuclear, then the $\sigma(E', E)$-closed convex circled hull of A is hypernuclear.*

(e) *If A is hypernuclear, then A is prenuclear and hence equicontinuous.*

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(f) *If B_0 is hypernuclear and $\{(B_n, \mu_n): n=1, 2 \cdots\}$ is a corresponding sequence, then each B_n is also hypernuclear.*

We remark here that the hypernuclear subsets of E' depend on the particular topology on E. Moreover, it follows from the theorem of Pietsch stated above that E is a nuclear space if and only if every equicontinuous subset of E' is hypernuclear.

DEFINITION. Let E be a locally convex space. A family $\mathfrak{F}$ of hypernuclear subsets of E' is said to be a full family if the following conditions hold.

(a) $\bigcup \mathfrak{F} = E'$.

(b) If $A, B \in \mathfrak{F}$, then $A \cup \lambda B \in \mathfrak{F}$ for all scalars λ.

(c) If $A \in \mathfrak{F}$ and $B \subset A$, then $B \in \mathfrak{F}$.

(d) If $A \in \mathfrak{F}$, then the $\sigma(E', E)$-closed convex circled hull of A belongs to $\mathfrak{F}$.

(e) If $B_0 \in \mathfrak{F}$, then there is at least one corresponding sequence $\{(B_n, \mu_n): n=1, 2, \cdots\}$ such that $B_n \in \mathfrak{F}$ for all n.

In the theorems below, it will be important to distinguish among different topologies on the same space. By (E, ρ), we shall mean that the space E is being considered with the topology ρ.

THEOREM 2. *Let (E, ρ) be a locally convex space, and let $\mathfrak{F}$ be a full family of ρ-hypernuclear subsets of E'. Let $\rho_{\mathfrak{F}}$ be the topology on E of uniform convergence on the members of $\mathfrak{F}$. Then $(E, \rho_{\mathfrak{F}})$ is a nuclear space having the same dual space as (E, ρ).*

PROOF. Since members of $\mathfrak{F}$ are all ρ-equicontinuous, we conclude that $(E, \rho_{\mathfrak{F}})' = (E, \rho)'$. The $\rho_{\mathfrak{F}}$-equicontinuous subsets of E' are just the members of $\mathfrak{F}$, and so it suffices, by Pietsch's result, to show that each $B_0 \in \mathfrak{F}$ is $\rho_{\mathfrak{F}}$-prenuclear. Let $\{(B_n, \mu_n): n=1, 2, \cdots\}$ be a ρ-corresponding sequence for B_0 such that $B_n \in \mathfrak{F}$ for all n. Now μ_1 is a $\sigma((E, \rho_{\mathfrak{F}})', E)$-Radon measure on the $\rho_{\mathfrak{F}}$-equicontinuous set B_1, and hence B_0 is also $\rho_{\mathfrak{F}}$-prenuclear.

In the following theorem, $\tau(E, F)$ will denote the Mackey topology on E, that is, the largest topology on E which is consistent with the duality $\langle E, F\rangle$.

THEOREM 3. *Let $\langle E, F\rangle$ be a duality. A topology ρ on E is consistent with the duality and makes E a nuclear space if and only if it is the topology of uniform convergence on some full family of $\tau(E, F)$-hypernuclear sets in F.*

PROOF. The "if" part is simply the previous theorem with $\rho = \tau(E, F)$.

Let ρ be a topology on E such that (E, ρ) is a nuclear space and $(E, \rho)' = F$. Then every member of the family $\mathfrak{F}$ of all ρ-equicontinuous subsets of F is ρ-hypernuclear. We claim that $\mathfrak{F}$ is a full family of $\tau(E, F)$-hypernuclear sets. Conditions (a)–(d) for $\mathfrak{F}$ are easily checked. Every ρ-equicontinuous set is $\tau(E, F)$-equicontinuous; and hence it follows that if $B_0 \in \mathfrak{F}$, then any ρ-corresponding sequence $\{(B_n, \mu_n): n = 1, 2, \cdots\}$ will also be $\tau(E, F)$-corresponding. Thus $\mathfrak{F}$ is also a full family of $\tau(E, F)$-hypernuclear sets.

It follows from the above results that the weak topology and the topology of uniform convergence on all $\tau(E, F)$-hypernuclear sets are respectively the weakest and strongest topologies on E which are consistent with the duality $\langle E, F\rangle$ and which make E a nuclear space.

We shall conclude with an example of a prenuclear set which is not hypernuclear.

EXAMPLE. Let I be an uncountable index set. Let $l_1(I)$ and $l_2(I)$ be the collections of all families $\{x_\alpha : \alpha \in I\}$ of scalars such that $\sum_I |x_\alpha| < \infty$ and $\sum_I |x_\alpha|^2 < \infty$, respectively. These spaces become Banach spaces under the norms $\|\{x_\alpha\}\|_1 = \sum_I |x_\alpha|$ and $\|\{x_\alpha\}\|_2 = (\sum_I |x_\alpha|^2)^{1/2}$.

The identity map $i: l_1(I) \to l_2(I)$ is absolutely summing [2, p. 39]; or equivalently [2, p. 36], there is a positive $\sigma(l_1(I)', l_1(I))$-Radon measure μ on the unit ball M of $l_1(I)'$ such that for each $x \in l_1(I)$

$$\|x\|_2 \leqq \int_M |\langle x, x'\rangle| \, d\mu(x').$$

Let B be the unit ball of $l_2(I)$, and consider B as a subset of $l_1(I)'$. Then for $y \in B$ and $x \in l_1(I)$,

$$\begin{aligned} |\langle x, y\rangle| &\leqq \|x\|_2 \|y\|_2 \leqq \|x\|_2 \\ &\leqq \int_M |\langle x, x'\rangle| \, d\mu(x'). \end{aligned}$$

Hence B is a prenuclear subset of $l_1(I)'$.

Now assume that B is a hypernuclear subset of $l_1(I)'$, and let ρ be the topology on $l_1(I)$ of uniform convergence on all hypernuclear subsets of $l_1(I)'$. Both of the injection maps j and k

$$(l_1(I), \| - \|_1) \xrightarrow{j} (l_1(I), \rho) \xrightarrow{k} (l_2(I), \| - \|_2)$$

can be seen to be continuous. By Theorem 2, $(l_1(I), \rho)$ is a nuclear space, and hence k is a nuclear map. Since the composition of a nuclear map with a continuous map is still nuclear, we conclude that $i = k \circ j$ is a nuclear map. But this is a contradiction since i is not even compact [2, p. 40]. Hence B cannot be hypernuclear.

References

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LOUISIANA STATE UNIVERSITY, BATON ROUGE