NUCLEAR TOPOLOGIES CONSISTENT WITH A DUALITY

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Let $E$ be a (Hausdorff) locally convex space, and let $E'$ be its dual space. A subset $B \subseteq E'$ is said to be prenuclear if there exist a $\sigma(E', E)$-closed equicontinuous subset $A \subseteq E'$ and a positive Radon measure $\mu$ on $A$ such that for each $x \in E$,

$$\sup_{y' \in B} |\langle x, y' \rangle| \leq \int_A |\langle x, x' \rangle| \, d\mu(x').$$

Pietsch has shown that $E$ is a nuclear space if and only if every equicontinuous subset of $E'$ is prenuclear (see [1] or [3]). We shall use this result to characterize all nuclear topologies on a locally convex space which are consistent with a given duality. We refer the reader to [3] for the basic results and notation that we shall use.

We begin with a definition.

DEFINITION. Let $E$ be a locally convex space, and let $B_0 \subseteq E'$. We shall say that $B_0$ is a hypernuclear set if there exists a sequence \( \{ (B_n, \mu_n) : n = 1, 2, \ldots \} \) where each $B_n$ is a $\sigma(E', E)$-closed equicontinuous subset of $E'$ and $\mu_n$ is a positive $\sigma(E', E)$-Radon measure on $B_n$ such that for each $n \geq 1$ and each $x \in E$,

$$\sup_{y' \in B_{n-1}} |\langle x, y' \rangle| \leq \int_{B_n} |\langle x, x' \rangle| \, d\mu_n(x').$$

We shall say that the sequence \( \{ (B_n, \mu_n) : n = 1, 2, \ldots \} \) corresponds to the hypernuclear set $B_0$.

In the following proposition, we state some elementary properties of hypernuclear sets. The proof is fairly direct, and so it has been omitted.

**Proposition 1.** (a) Finite subsets of $E'$ are hypernuclear.
(b) If $A$ and $B$ are hypernuclear and $\lambda$ is a scalar, then $A \cup \lambda B$ is hypernuclear.
(c) If $A$ is hypernuclear and $B \subseteq A$, then $B$ is hypernuclear.
(d) If $A$ is hypernuclear, then the $\sigma(E', E)$-closed convex circled hull of $A$ is hypernuclear.
(e) If $A$ is hypernuclear, then $A$ is prenuclear and hence equicontinuous.

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565
(f) If $B_0$ is hypernuclear and $\{(B_n, \mu_n): n = 1, 2 \cdots\}$ is a corresponding sequence, then each $B_n$ is also hypernuclear.

We remark here that the hypernuclear subsets of $E'$ depend on the particular topology on $E$. Moreover, it follows from the theorem of Pietsch stated above that $E$ is a nuclear space if and only if every equicontinuous subset of $E'$ is hypernuclear.

**Definition.** Let $E$ be a locally convex space. A family $\mathcal{F}$ of hypernuclear subsets of $E'$ is said to be a full family if the following conditions hold.

(a) $\bigcup \mathcal{F} = E'$.
(b) If $A, B \in \mathcal{F}$, then $A \cup \lambda B \in \mathcal{F}$ for all scalars $\lambda$.
(c) If $A \in \mathcal{F}$ and $B \subseteq A$, then $B \in \mathcal{F}$.
(d) If $A \in \mathcal{F}$, then the $\sigma(E', E)$-closed convex circled hull of $A$ belongs to $\mathcal{F}$.
(e) If $B_0 \in \mathcal{F}$, then there is at least one corresponding sequence $\{(B_n, \mu_n): n = 1, 2, \cdots\}$ such that $B_n \in \mathcal{F}$ for all $n$.

In the theorems below, it will be important to distinguish among different topologies on the same space. By $(E, \rho)$, we shall mean that the space $E$ is being considered with the topology $\rho$.

**Theorem 2.** Let $(E, \rho)$ be a locally convex space, and let $\mathcal{F}$ be a full family of $\rho$-hypernuclear subsets of $E'$. Let $\rho_\mathcal{F}$ be the topology on $E$ of uniform convergence on the members of $\mathcal{F}$. Then $(E, \rho_\mathcal{F})$ is a nuclear space having the same dual space as $(E, \rho)$.

**Proof.** Since members of $\mathcal{F}$ are all $\rho$-equicontinuous, we conclude that $(E, \rho_\mathcal{F})' = (E, \rho)'$. The $\rho_\mathcal{F}$-equicontinuous subsets of $E'$ are just the members of $\mathcal{F}$, and so it suffices, by Pietsch's result, to show that each $B_0 \in \mathcal{F}$ is $\rho_\mathcal{F}$-prenuclear. Let $\{(B_n, \mu_n): n = 1, 2, \cdots\}$ be a $\rho$-corresponding sequence for $B_0$ such that $B_n \in \mathcal{F}$ for all $n$. Now $\mu_1$ is a $\sigma((E, \rho_\mathcal{F})', E)$-Radon measure on the $\rho_\mathcal{F}$-equicontinuous set $B_1$, and hence $B_0$ is also $\rho_\mathcal{F}$-prenuclear.

In the following theorem, $\tau(E, F)$ will denote the Mackey topology on $E$, that is, the largest topology on $E$ which is consistent with the duality $\langle E, F \rangle$.

**Theorem 3.** Let $\langle E, F \rangle$ be a duality. A topology $\rho$ on $E$ is consistent with the duality and makes $E$ a nuclear space if and only if it is the topology of uniform convergence on some full family of $\tau(E, F)$-hypernuclear sets in $F$.

**Proof.** The "if" part is simply the previous theorem with $\rho = \tau(E, F)$.
Let $\rho$ be a topology on $E$ such that $(E, \rho)$ is a nuclear space and $(E, \rho)' = F$. Then every member of the family $\mathcal{F}$ of all $\rho$-equicontinuous subsets of $F$ is $\rho$-hypernuclear. We claim that $\mathcal{F}$ is a full family of $\tau(E, F)$-hypernuclear sets. Conditions (a)–(d) for $\mathcal{F}$ are easily checked. Every $\rho$-equicontinuous set is $\tau(E, F)$-equicontinuous; and hence it follows that if $B_0 \subseteq \mathcal{F}$, then any $\rho$-corresponding sequence $\{(B_n, \mu_n): n = 1, 2, \ldots\}$ will also be $\tau(E, F)$-corresponding. Thus $\mathcal{F}$ is also a full family of $\tau(E, F)$-hypernuclear sets.

It follows from the above results that the weak topology and the topology of uniform convergence on all $\tau(E, F)$-hypernuclear sets are respectively the weakest and strongest topologies on $E$ which are consistent with the duality $\langle E, F \rangle$ and which make $E$ a nuclear space.

We shall conclude with an example of a prenuclear set which is not hypernuclear.

**Example.** Let $I$ be an uncountable index set. Let $l_1(I)$ and $l_2(I)$ be the collections of all families $\{x_\alpha: \alpha \in I\}$ of scalars such that $\sum_I |x_\alpha| < \infty$ and $\sum_I |x_\alpha|^2 < \infty$, respectively. These spaces become Banach spaces under the norms $\|\{x_\alpha\}\|_1 = \sum_I |x_\alpha|$ and $\|\{x_\alpha\}\|_2 = (\sum_I |x_\alpha|^2)^{1/2}$.

The identity map $i: l_1(I) \to l_2(I)$ is absolutely summing [2, p. 39]; or equivalently [2, p. 36], there is a positive $\sigma(l_1(I)', l_1(I))$-Radon measure $\mu$ on the unit ball $M$ of $l_1(I)'$ such that for each $x \in l_1(I)$

$$\|x\|_2 \leq \int_M |\langle x, x' \rangle| \, d\mu(x').$$

Let $B$ be the unit ball of $l_2(I)$, and consider $B$ as a subset of $l_1(I)'$. Then for $y \in B$ and $x \in l_1(I)$,

$$|\langle x, y \rangle| \leq \|x\|_2 \|y\|_2 \leq \|x\|_2 \leq \int_M |\langle x, x' \rangle| \, d\mu(x').$$

Hence $B$ is a prenuclear subset of $l_1(I)'$.

Now assume that $B$ is a hypernuclear subset of $l_1(I)'$, and let $\rho$ be the topology on $l_1(I)$ of uniform convergence on all hypernuclear subsets of $l_1(I)'$. Both of the injection maps $j$ and $k$

$$(l_1(I), \| - \|_1) \xrightarrow{j} (l_1(I), \rho) \xrightarrow{k} (l_2(I), \| - \|_2)$$

can be seen to be continuous. By Theorem 2, $(l_1(I), \rho)$ is a nuclear space, and hence $k$ is a nuclear map. Since the composition of a nu-
clear map with a continuous map is still nuclear, we conclude that $i = k \circ j$ is a nuclear map. But this is a contradiction since $i$ is not even compact [2, p. 40]. Hence $B$ cannot be hypernuclear.

REFERENCES


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