REPLACING CERTAIN MAPS OF 3-MANIFOLDS BY HOMEOMORPHISMS

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Let f be a map of a 3-manifold M_1 onto the 3-manifold M_2. Call f boundary preserving if $f|$ Bd M_1 is a homeomorphism onto Bd M_2 and $f(\text{Int } M_1) = \text{Int } M_2$. Let $S_f = \{x | x \in M_1, \text{ and } f^{-1}f(x) \text{ is nondegenerate}\}$. In [3], McMillan announced that if $M_1 = S^3$ and Cl $f(S_f)$ is 0-dimensional, then $M_2 = S^3$. This follows by showing $S^3 - f^{-1}(x) = E^3$ (f induces a point-like decomposition of S^3). By Bing's example in [1, p. 7], M_2 may be homeomorphic to M_1 even though for certain points x, $S^3 - f^{-1}(x) \neq E^3$. In this note we wish to develop a condition on Cl $f(S_f)$ which will enable us to say M_1 and M_2 are homeomorphic.

In Theorem 5 of [2], Hempel showed how to obtain a boundary preserving map of a cube with one knotted hole onto a cube with one handle; however, in this case, the injection of $H_1(\text{Cl } f(S_f); Z)$ into $H_1(M_2; Z) = Z$ is nontrivial. This fact together with Waldhausen's Theorem 1.2 of [5] suggest the type of condition on Cl $f(S_f)$ we are looking for. Following the definition given in [3], we say X, a subset of the interior of a 3-manifold M, is strongly n-acyclic (over Z) if for each open set $U \subset M$ such that $X \subset U$ there is an open set V such that $X \subset V \subset U$ and injection of $H_n(V; Z)$ into $H_n(U; Z)$ is trivial.

THEOREM 1. *Suppose M_1 is a compact, connected, irreducible 3-manifold with nonempty boundary, and M_2 is an orientable irreducible 3-manifold. Then if f is a boundary preserving map of M_1 onto M_2 and there is a closed set X so that $f(S_f) \subset X \subset \text{Int } M_2$ and each component of X is strongly 1-acyclic, then M_1 is homeomorphic to M_2.*

PROOF. If M_1 is a 3-cell we are finished. If not, there is a polyhedral orientable (2-sided) surface F_1 so that $0 \neq [\partial F_1] \in H_1(\partial M_2, Z)$ and Int $F_1 \subset \text{Int } M_2$. There is a compact 3-manifold V_1 so that $X \subset \text{Int } V_1 \subset \text{Int } M_2$ and injection $H_1(V_1, Z)$ in $H_1(M_2, Z)$ is trivial. Let G be a bouquet of simple closed curves in some component C of Bd V_1 such that $C - G$ is an open 2-cell. Assume G intersects and pierces F_1 a finite number of times away from its vertex. By hypothesis each loop of G is null homotopic in Int M_2, hence it follows that there is an arc α in G such that Int α does not contain the vertex of G or any point of F_1 and α starts and ends on the same side of F_1. Replace small disks

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in F_1 about the endpoints of α by an annulus running close to α to obtain an orientable (2-sided) surface F_1' with fewer points of intersection with G than F_1. It follows that we may assume $F_1 \cap V_1 = \emptyset$. From $f^{-1}(F_1)$ we may obtain the incompressible surface G_1 so that Bd $G_1 =$ Bd $f^{-1}(F_1)$ and there is a collection of disjoint disks $D_{11}, \cdots, D_{1n(1)}$ in G_1 such that $(G_1 - \bigcup_{i=1}^{n(1)} \text{Int } D_{1i}) \cap \text{Cl } S_f = \emptyset$. (See p. 58 of [5] for a definition of incompressible surface. As in the proof of Lemma 1.1, Part (3) of [6], the Loop Theorem and Dehn's Lemma are used to reduce $f^{-1}(F_1)$ to an incompressible surface; hence it is important that $f^{-1}(F_1)$ is 2-sided since M_1 was not assumed to be orientable.) By Dehn's Lemma there are disjoint disks $E_{11}, \cdots, E_{1n(1)}$ so that $G_1' = f(G_1 - (\bigcup_{i=1}^{n(1)} \text{Int } D_{1i})) \cup (\bigcup_{i=1}^{n(1)} E_{1i})$ is a surface in M_2 homeomorphic to G_1. Let $M_2(1) = M_2$ minus a regular neighborhood $U(G_1') = G_1' \times (-1, 1)$ of G_1'. If Bd $M_2(1)$ is not a 2-sphere, there is a surface F_2 so that $0 \neq [\partial F_2] \in H_1(\partial M_2(1), Z)$ and each $E_{1i} \times \{-1, 1\} \cap \text{Bd } F_2 = \emptyset$. Starting with F_2 we now repeat the same process as was carried out with F_1. That is, obtain from $f^{-1}(F_2)$ the incompressible surface G_2 so that Bd $G_2 =$ Bd $f^{-1}(F_2)$ and there is a collection of disjoint disks $D_{21}, \cdots, D_{2n(2)}$ in G_2 such that $(G_2 - \bigcup_{i=1}^{n(2)} \text{Int } D_{2i}) \cap \text{Cl } S_f = \emptyset$ and, in addition, cut G_2 off on $\bigcup_{i=1}^{n(1)} \text{Int } D_{1i}$ so as to assume Int $G_2 \cap G_1 = \emptyset$. Since Proposition 2.4 of [5] holds for the resulting sequence $G_1, \cdots, G_i$ (even though M_1 may not be orientable), there is an n such that by splitting M_1 successively along the $G_1, \cdots, G_n$ the remaining component is a 3-cell B. It follows by adding a regular neighborhood of each $D_{11}, \cdots, D_{1n(1)}, D_{21}, \cdots, D_{2n(2)}, \cdots, D_{m(1)}, \cdots, D_{mn(m)}$ to B that Cl S_f is contained in a 3-manifold Q which is a cube with handles (with orientable handles since Bd Q is orientable). Since f is a homeomorphism of $M_1 -$ Int Q onto $M_2 - f(Q)$, we may use Dehn's Lemma and the irreducibility of M_2 to replace f by a homeomorphism h of Q onto $f(Q)$ agreeing with f on Bd Q. Hence M_1 is homeomorphic to M_2.

COROLLARY 1. *Suppose M_1 is an irreducible 3-manifold with nonempty boundary. Then if f: $M_1 \to M_2$ is boundary preserving and $f(\text{Cl } S_f)$ is 0-dimensional, then M_1 is homeomorphic to M_2.*

In particular, we have the result announced by McMillan in [3] that if f is a map of S^3 onto a 3-manifold M_2 and $f(\text{Cl } S_f)$ is 0-dimensional, then S^3 is homeomorphic to M_2 and, in addition, for each $x \in M_2$, $S^3 - f^{-1}(x)$ is homeomorphic to E^3.

If we further restrict the type of 3-manifold that M_1 and M_2 can be, it may be possible to change the requirement on $f(S_f)$. For instance, we have the following.

THEOREM 2. *Suppose M_1, M_2 are cubes with one hole and the commutator subgroup of $\pi_1(M_1)$ is finitely generated. Then if f is a boundary preserving map of M_1 onto M_2 and $f(\mathrm{Cl}\ S_f)$ shrinks to a point in M_2, then M_1 is homeomorphic to M_2.*

PROOF. There is an orientable surface F_1 so that Bd F_1 is a nonseparating simple closed curve in Bd M_2 and Int $F_1 \subset$ Int M_2. Since $f(\mathrm{Cl}\ S_f)$ shrinks to a point in M_2 we may, as in Theorem 1, assume $F_1 \cap f(\mathrm{Cl}\ S_f) = \emptyset$. Let G_1 be the incompressible surface in M_1 obtained from $f^{-1}(F_1)$ and assume $D_1, \cdots, D_n$ are disjoint disks on G_1 so that $(G_1 - \bigcup_{i=1}^n \mathrm{Int}\ D_i) \cap \mathrm{Cl}\ S_f = \emptyset$. Since the commutator subgroup of $\pi_1(M_1)$ is finitely generated, by the proof of Theorem 1 of [4] it follows that M_1 minus an open regular neighborhood of G_1 in M_1 is a cube with handles; hence it follows that Cl S_f lies in a cube with handles and M_1 is homeomorphic to M_2.

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