# OPERATORS COMMUTING WITH A WEIGHTED SHIFT

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1. **Introduction.** We say that the sequence of vectors $\{y_n\}_{-\infty}^{\infty}$ in a separable infinite-dimensional complex Banach space $\boldsymbol{B}$ is a Schauder basis if for each element $f$ in $\boldsymbol{B}$ there is a unique sequence $\{c_n\}_{-\infty}^{\infty}$ of scalars such that $f=\sum_{-\infty}^{\infty} c_n y_n$ where the limit (in norm) is taken independently in the positive and negative directions.

Given such a basis with $\|y_n\|=1$ for all $n$, let $\boldsymbol{a}=\{a_n\}_{-\infty}^{\infty}$ be a sequence of nonzero scalars such that the linear operator $T=T_{\boldsymbol{a}}$ defined by $T\sum c_n y_n=\sum c_n a_{n+1} y_{n+1}$ is bounded. (A necessary condition on $\boldsymbol{a}$ is that $\sup|a_n|<\infty$.) $T$ is called a (weighted) shift operator.

The present paper defines an isomorphism between the ring of operators commuting with $T$ and a ring of formal "Laurent" series. This leads to conclusions concerning the solvability of certain equations in $T$, including those defining the spectrum. Under reasonable additional conditions the spectrum of $T$ is a zero-centered annulus (or a disc) with radii a function of $\boldsymbol{a}$ (Theorem 10). When this annulus has nonempty interior, $T$ has no reducing subspaces and no $n$th roots (Corollary 2). In Theorem 9 a large collection of operators without roots is exhibited.

The parts of the spectrum of $T$, cyclic vectors, and some material on invariant subspaces will be the subject of a separate paper [7].

2. **Representation.** We shall endow $\boldsymbol{B}$ with the structure of a Banach space whose elements are formal "Laurent" series. To each vector $f=\sum c_n y_n$ in $\boldsymbol{B}$ we associate the series $f(z)=\sum_{-\infty}^{\infty} b_n z^n$ where $b_n=c_n(\prod_{i=1}^{n} a_i)^{-1}$ $(n>0)$, $b_0=c_0$, $b_n=c_n(\prod_{i=n+1}^{0} a_i)$ $(n<0)$. Identifying the vector $f$ and its associated series by writing $f=f(z)$ will cause no confusion. The sequence of elements $\{z^n\}_{-\infty}^{\infty}$ forms a new Schauder basis for $\boldsymbol{B}$ because each $z^n$ is a nonzero scalar multiple of $y_n$.

For $f(z)=\sum b_n z^n$ we define $(f)_n=b_n$. The mappings $f\to(f)_n$ are bounded linear functionals on $\boldsymbol{B}$; the boundedness is established in [1, p. 111], where it is shown that, in fact, there exists a number $M>0$ depending on the basis such that

$$|(f)_n| \leqq M\|f\|/\|z^n\| \quad \text{for all } n. \tag{2.1}$$

These mappings are total in the sense that if $(f)_n=(g)_n$ for all $n$, then $f=g$.

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Note that $\|z^n\| = |\prod_{i=1}^{n} a_i|$ $(n>0)$, $\|1\| = \|z^0\| = 1$, and $\|z^n\| = |\prod_{i=n+1}^{0} a_i|^{-1}$ $(n<0)$. It is easy to check that $Tf(z) = zf(z)$ for all $f$ in $\boldsymbol{B}$.

## 3. Commuting operators.

THEOREM 1. *Suppose $f(z) = \sum b_n z^n \in \boldsymbol{B}$ has the property that for all $g(z) = \sum c_n z^n \in \boldsymbol{B}$ the formal product*

$$f(z)g(z) = \sum_{i=-\infty}^{\infty} \left( \sum_{j=-\infty}^{\infty} b_{i-j} c_j \right) z^i$$

*lies in $\boldsymbol{B}$ (where the inner sum is required to converge in the usual sense). Then the linear transformation $S$ defined by the equation $S(g(z)) = f(z)g(z)$ is bounded.*

PROOF. $(S(g(z)))_i = (f(z)g(z))_i = \sum_{j=-\infty}^{\infty} b_{i-j}c_j$. Thus $S$ can be defined by the doubly infinite matrix $(b_{i-j})$ with respect to the basis formed by the sequence $\{z^n\}_{-\infty}^{\infty}$. Now we simply exploit the fact that a Banach space mapping defined everywhere by a matrix with respect to a Schauder basis is bounded [2, Theorem 1, p. 689].

Next we present a converse to the above theorem.

THEOREM 2. *If the bounded linear operator $S$ commutes with $T$, then $S$ is multiplication by the series $f(z) = S(1) = S(z^0)$. In other words $Sg(z) = S(1) \cdot g(z) = f(z)g(z)$ for all $g \in \boldsymbol{B}$.*

PROOF. Let $S(1) = f(z) = \sum b_n z^n$ and let $g(z) = \sum c_n z^n$. For $j \geqq 0$, $(S(z^j))_i = (ST^j(1))_i = (T^jS(1))_i = b_{i-j}$. For $j<0$, $(S(z^j))_i = (T^{-j}S(z^j))_{i-j} = (ST^{-j}(z^j))_{i-j} = (S(1))_{i-j} = b_{i-j}$ as before. Hence

$$(Sg(z))_i = \left( S\left( \sum_j c_j z^j \right) \right)_i = \sum_j c_j (S(z^j))_i = \sum_{j=-\infty}^{\infty} b_{i-j}c_j = (f(z)g(z))_i.$$

Thus $Sg(z) = f(z)g(z)$.

Define $\mathfrak{F}$ to be the set of "Laurent" series $f(z)$ in $\boldsymbol{B}$ such that multiplication by $f(z)$ in $\boldsymbol{B}$ is a bounded operator, which we shall denote $f(T)$. Theorems 1 and 2 show that under the natural algebraic operations and the norm $\|f(z)\|_{\mathfrak{F}} = \|f(T)\|$, $\mathfrak{F}$ is a Banach algebra.

COROLLARY 1. *Two operators commuting with $T$ commute with each other.*

PROOF. Laurent series multiplication is commutative.

## 4. Analytic behavior in $\mathfrak{F}$.

THEOREM 3. (1) *The limits*

$$R_1 = \lim_{n\to\infty}\left(\sup_m \prod_{i=m+1}^{m+n} |a_i|\right)^{1/n}$$

*and*

$$R_2 = \lim_{n\to\infty}\left(\inf_m \prod_{i=m+1}^{m+n} |a_i|\right)^{1/n}$$

*exist for any bounded sequence* $\mathbf{a}=\{a_n\}_{-\infty}^{\infty}$. (*Note that if* $\mathbf{a}$ *contains no zero terms* $\prod_{i=m+1}^{m+n}|a_i| = \|z^{m+n}\|/\|z^m\|$.)

(2) *In the particular case that* $\boldsymbol{B}$ *is a Hilbert space and* $\{y_n\}_{-\infty}^{\infty}$ *is an orthonormal basis,* (a) $R_1$ *is the spectral radius of* $T_{\mathbf{a}}$; (b) *The conditions* (i) $T_{\mathbf{a}}^{-1}$ *exists,* (ii) $\inf|a_n|>0$, *and* (iii) $R_2>0$ *are all equivalent. When these conditions are satisfied,* $R_2^{-1}$ *is the spectral radius of* $T_{\mathbf{a}}^{-1}$.

PROOF. (1) Clearly if $\inf|a_n|=0$ then $R_2=0$. The existence of the limits in other cases follows a fortiori from (2).

(2) It is easy to check that, for $n>0$, $\|T^n\|=\sup_m\prod_{i=m+1}^{m+n}|a_i|$, which implies (a). The equivalence of (i) and (ii) is clear. When $T_{\mathbf{a}}^{-1}$ exists it is easy to check that, for $n>0$, $\|T_{\mathbf{a}}^{-n}\|=\sup_m\prod_{i=m+1}^{m+n}|a_i|^{-1}$ $=(\inf_m\prod_{i=m+1}^{m+n}|a_i|)^{-1}$, and then the rest of (b) follows without difficulty.

For $f(z)=\sum b_nz^n$ in $\boldsymbol{B}$ define $f^+(z)=\sum_0^\infty b_nz^n$ and $f^-(z)=\sum_{-\infty}^{-1} b_nz^n$.

THEOREM 4. *If* $f(z)\in\mathfrak{T}$ (*that is, if* $f(T)$ *is a bounded operator*), *then*:

(1) *If* $R_1\neq 0$, $f^+(z)$ *converges to an analytic function on the open disc* $|z|<R_1$.

(2) *If* $R_2=0$, $f^-(z)\equiv 0$.

(3) *If* $R_2\neq 0$, $f^-(z)$ *converges to an analytic function in the region* $|z|>R_2$.

PROOF. $\|f(T)\| \geqq \|f(z)z^m\|/\|z^m\| = \|\sum b_nz^{n+m}\|/\|z^m\| \geqq |b_n|\,\|z^{n+m}\|/M\|z^m\|$, where we use (2.1) at the last step. It follows that

$$\begin{aligned} |b_n| &\leqq \left(\inf_m \|z^m\|/\|z^{n+m}\|\right) M\|f(T)\| \\ &= \left(\sup_m \|z^{n+m}\|/\|z^m\|\right)^{-1} M\|f(T)\|. \end{aligned} \tag{4.1}$$

Taking the $n$th root of both sides we immediately obtain $\limsup_{n\to\infty}|b_n|^{1/n}$ $\leqq R_1^{-1}$, which proves (1). Theorem 3 shows that the equality $R_2=0$ implies that $\inf_m\|z^{m+n}\|/\|z^m\|=0$ for $n>0$. Now using (4.1), for $n>0$ we obtain

$$|b_{-n}| \leqq \left(\inf_m \|z^m\|/\|z^{m-n}\|\right) M\|f(T)\|$$

$$= \left(\inf_m \|z^{m+n}\|/\|z^m\|\right) M\|f(T)\| = 0$$

if $R_2=0$, and then $f^-(z)\equiv 0$, which proves (2).

If $R_2\neq 0$, by taking the $n$th root in the above inequality we obtain $\limsup_{n\to\infty} |b_{-n}|^{1/n}\leqq R_2$, which is (3).

Now let $D=D_a$ be the closed set $R_2\leqq |z| \leqq R_1$. *If $R_1\neq R_2$, Theorems 2 and 4 tell us that every operator commuting with $T$ is multiplication in* ***B*** *by a Laurent series converging to a function analytic in the interior of $D$.*

COROLLARY 2. *If $R_1\neq R_2$,*

(1) *$T$ commutes with no projection except zero and the identity, and*

(2) *$T$ has no roots.*

*In fact,*

(3) *if $|\lambda| < R_1$, then $T-\lambda$ has no roots.*

*The same conclusions hold if $R_1=R_2=0$.*

PROOF. When $R_1\neq R_2$

(1) if $(f(T))^2=f(T)$ then $f(z)(f(z)-1)=0$ on the interior of $D$. But the product of analytic functions does not vanish identically unless one of them does so.

(3) includes (2). If $(f(T))^n=T-\lambda$, then $(f(z))^n\equiv z-\lambda$ on the interior of $D$. But the function $z-\lambda$ has no analytic $n$th roots on $D$ if $|\lambda| < R_1$.

When $R_1=R_2=0$ we have an analogous proof. The identities above become identities of formal *power* series rather than of analytic functions. But since the product of two power series is zero only if one of the factors is zero, and since no $n$th power of a power series can be the series $z$, the conclusions still hold.

There has been a certain amount of interest in invertible operators without roots. The first example of such operators was given by Halmos, Lumer and Schäffer [**3**]. In [**5**], Schäffer showed that if $\{y_n\}_{-\infty}^{\infty}$ is an orthonormal basis in Hilbert space and if $\limsup_{n\to-\infty}|a_n| < \liminf_{n\to\infty}|a_n|$ (and $0<\inf_n|a_n| \leqq \sup_n|a_n| < \infty$), then $T$ is in the interior (in the sense of the norm topology) of the set of invertible operators without roots. Corollary 2 above provides a large class of rootless operators. An even larger class will be given in Theorem 9. Invertibility is the next topic.

## 5. The spectrum of $T$.

THEOREM 5. *A spectral mapping theorem. If $R_1 \neq R_2$ and if $f \in \mathfrak{F}$, then spectrum $f(T) \supseteq \{f(\lambda) \mid \lambda \in \text{interior } D\}$.*

PROOF. Suppose $f(\lambda) \notin \text{spectrum } f(T)$. Let $g(z) = (f(\lambda) - f(T))^{-1}(1)$. Then $g(z)(f(\lambda) - f(z)) \equiv 1$ on the interior of $D$. Letting $z = \lambda$ leads to a contradiction.

It follows immediately from this theorem that if $R_1 \neq R_2$ and if $f \in \mathfrak{F}$ then $f(z)$ is bounded on the interior of $D$ by the constant $\|f(T)\|$. Thus convergence in $\mathfrak{F}$ implies uniform convergence on the interior of $D$.

THEOREM 6. *$D \subseteq$ spectrum $T$.*

PROOF. We need only consider the case where $R_1 = R_2$, for the case $R_2 < R_1$ follows immediately from Theorem 5. When $R_1 = R_2 = 0$, Theorem 4, part (2) tells the story. Otherwise, when $(\lambda - T)^{-1}$ exists, $(\lambda - z)(\lambda - T)^{-1}(1) = 1$ as formal "Laurent" series. Theorem 4 and simple computation show that $(\lambda - T)^{-1}(1)$ can only be $\sum_0^\infty \lambda^{-n-1} z^n$ if $|\lambda| > R_2$, and can only be $\sum_{-\infty}^{-1} -\lambda^{-n-1} z^n$ if $0 < |\lambda| < R_1$. Therefore, if $|\lambda_0| = R_1 = R_2$, $\lambda_0$ is not a point of continuity of $(\lambda - T)^{-1}$, but every point not in spectrum $T$ *is* a point of continuity of $(\lambda - T)^{-1}$ [**4**, p. 416].

Theorem 2 developed a functional calculus for operators commuting with $T$. The next theorem shows that this functional calculus fits nicely into the usual functional calculus [**4**, p. 431].

THEOREM 7. *Let $f(z)$ be a function analytic on a neighborhood of the spectrum of $T$. Then $f(z) \in \mathfrak{F}$. The operator $f(T)$ defined by the usual functional calculus is the same as $f(T)$ defined in the remark after Theorem 2.*

PROOF. First note that $f(z)$ is analytic on a neighborhood of $D$, and so has a Laurent series there, and the statement $f(z) \in \mathfrak{F}$ makes sense. Since the operator $f(T)$ defined by the usual calculus commutes with $T$, Theorem 2 shows that it suffices to prove $f(T)(1) = f(z)$. In the proof of Theorem 6 we checked this formula for functions of the form $f(z) = (\lambda - z)^{-1}$, since for these functions the usual functional calculus defines $f(T) = (\lambda - T)^{-1}$. In general the usual functional calculus defines

$$f(T) = \frac{1}{2\pi i} \int_C f(\lambda)(\lambda - T)^{-1} d\lambda.$$

The curve $C$ is chosen so that we certainly have

$$f(z) \equiv \frac{1}{2\pi i}\int_C f(\lambda)(\lambda - z)^{-1}d\lambda$$

for all $z$ in $D$ and thus

$$(f(z))_j = \frac{1}{2\pi i}\int_C f(\lambda)((\lambda - z)^{-1})_j d\lambda$$

(where $(\ )_j$ here denotes the $j$th Laurent coefficient of the function). Thus

$$(f(T)(1))_j = \frac{1}{2\pi i}\int_C f(\lambda)((\lambda - T)^{-1}(1))_j d\lambda$$

$$= \frac{1}{2\pi i}\int_C f(\lambda)((\lambda - z)^{-1})_j = (f(z))_j$$

so $f(T)(1) = f(z)$.

THEOREM 8. *Spectrum $T$ is connected.*

PROOF. Otherwise [4, p. 421] for a properly chosen closed curve $C$, $P = \int_C (\lambda - T)^{-1}d\lambda$ would be a nontrivial projection commuting with $T$. But as in Theorem 7, we can calculate for $j > 0$,

$$(P(1))_j = \int_C ((\lambda - z)^{-1})_j d\lambda = \int_{\lambda\in C;\,|\lambda|>R_1} \lambda^{-j-1}d\lambda = 0.$$

Similarly $(P(1))_j = 0$ for $j < 0$. Thus $P$ must be trivial.

CONJECTURE. Spectrum $T$ always equals $D$. Note that Theorem 3, part (2), combined with Theorem 6 proves the conjecture for $\{y_n\}_{-\infty}^{\infty}$ an orthonormal basis in Hilbert space. In Theorem 10 we show that the conjecture is true for many familiar spaces. But first let us reveal some additional rootless operators.

THEOREM 9. *Suppose $R_1 \neq R_2$, $f(z)$ analytic and univalent on a neighborhood of spectrum $T$, and $f(z)$ has no single-valued analytic nth roots defined on the interior of $D$. Then $f(T)$ has no nth root.*

PROOF. By the usual spectral mapping theorem the function $g$ inverse to $f$ is analytic on a neighborhood of spectrum $f(T)$. In [4, p. 433] it is shown that $g(f(z)) \equiv z$ implies that $g(f(T)) = T$. Now if $S^n = f(T)$, then $S$ commutes with $f(T)$, and so $S$ commutes with $g(f(T)) = T$. Thus by Theorem 2, $S = h(T)$, where $h \in \mathfrak{F}$, and the conclusion follows as in Corollary 2.

## 6. A condition assuring that spectrum $T=D$.

THEOREM 10. *Let $\{y_n\}_{-\infty}^{\infty}$ be a fixed basis of a Banach space $\boldsymbol{B}$ with $\|y_n\|=1$ for all $n$. Suppose:*

*(1) for each bounded sequence $\boldsymbol{a}$ the operator $T_{\boldsymbol{a}}$ is bounded.*

*(2) if $\mathbf{1}$ is the sequence of all $\mathbf{1}$'s, spectrum $T_{\mathbf{1}}$ is the unit circle. Then if $\boldsymbol{a}$ is bounded, spectrum $T_{\boldsymbol{a}}=D_{\boldsymbol{a}}$. (We shall call a basis satisfying condition (1) an $N$-basis.)*

PROOF. We need the following

LEMMA. *The set of shift operators $\mathfrak{J}=\{T_{\boldsymbol{a}} \mid T_{\boldsymbol{a}} \text{ bounded}\}$ (where $\boldsymbol{a}$ may contain zero terms) is a subspace of the space of all bounded operators on $\boldsymbol{B}$ and is closed in the weak operator topology.*

PROOF. It is clear that $\mathfrak{J}$ is a linear manifold. Suppose $I$ is a directed set, and $\boldsymbol{a}_i=\{a_{ij}\}_{j=-\infty}^{\infty}$, $i\in I$, are sequences such that $T_{\boldsymbol{a}_i}$ are bounded operators converging in the weak operator topology to $T$. If $f\sum c_n y_n$ define $[f]_n=c_n$. For any $f\in\boldsymbol{B}$, $[Tf]_n=\lim_i [T_{\boldsymbol{a}_i}f]_n=\lim_i(c_{n-1}a_{in})$ $=c_{n-1}a_n$, where $a_n=\lim_i a_{in}$. Thus $Tf=\sum c_n a_{n+1}y_{n+1}$ and $T\in\mathfrak{J}$.

Now we prove Theorem 10. Let $\|\boldsymbol{a}\|_\infty$ be the sup norm. We know that $\|T_{\boldsymbol{a}}\|\geqq\|\boldsymbol{a}\|_\infty$ whenever $T_{\boldsymbol{a}}$ is bounded. Thus the map from $\mathfrak{J}$ to the Banach space of all bounded sequences sending $T_{\boldsymbol{a}}$ to $\boldsymbol{a}$ is norm-decreasing.

If $T_{\boldsymbol{a}}$ is bounded for every bounded $\boldsymbol{a}$, the map is 1-1 and onto and by [1, p. 41] it is bicontinuous. We conclude that there exists a constant $N$ such that $\|T_{\boldsymbol{a}}\|\leqq N\|\boldsymbol{a}\|_\infty$ for all bounded sequences $\boldsymbol{a}$.

For any bounded sequence $\boldsymbol{a}=\{a_k\}$ we now define the sequences $\boldsymbol{a}_n=\{a_{nj}\}$, $n>0$, by $a_{nj}=\prod_{k=j}^{n+j-1}a_k$. Since $(T_{\boldsymbol{a}})^n y_j=(\prod_{k=j+1}^{j+n}a_k)y_{j+n}$, we obtain $(T_{\boldsymbol{a}})^n=(T_1)^{n-1}(T_{\boldsymbol{a}_n})$. It follows that

$$\sup_j \prod_{k=j+1}^{j+n}|a_k| \leqq \|(T_{\boldsymbol{a}})^n\| \leqq \|(T_1)^{n-1}\|\,\|T_{\boldsymbol{a}_n}\| \tag{6.1}$$
$$\leqq \|(T_1)^{n-1}\| N \sup_j \prod_{k=j+1}^{j+n}|a_k|.$$

Taking the $n$th root and passing to the limit we find

$$R_1=\lim_{n\to\infty}\|T_{\boldsymbol{a}}^n\|\,1/n=\text{spectral radius } T_{\boldsymbol{a}}.$$

(Since $\lim_{n\to\infty}\|(T_1)^{n-1}\|^{1/n}=$spectral radius $T_1=1$.)

Since $D_{\boldsymbol{a}}\subseteq$spectrum $T_{\boldsymbol{a}}$, the proof will be complete if we can show that the spectral radius of $(T_{\boldsymbol{a}})^{-1}=R_2^{-1}$ whenever $R_2\neq 0$. But this is the statement for left shifts analogous to what we have just proven for right shifts. Thus it suffices to check the hypotheses for left shifts.

(1) For every bounded sequence $\mathbf{a}$, $T_{\mathbf{a}}T_1^{-2}$ is the left shift defined by the sequence and is bounded.

(2) $T_1^{-1}$ has the unit circle as its spectrum.

Finally we note that everything in this paper remains true for one-sided shifts (with obvious modifications).

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