

ON GENERATING SYSTEMS FOR ABELIAN GROUPS

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The purpose of this note is to investigate two kinds of generating sets for an abelian group. It is well known that if an abelian group has a projective cover then it has a very special kind of generating set: a free basis. The idea of a projective cover of an abelian group is dual to that of a divisible hull when the divisible hull is defined as an essential injective extension. This note considers the dualization of two other equivalent definitions of a divisible hull. This yields the definitions of a q -cover and an i -cover. It is shown that an abelian group has a q -cover if and only if it has a minimum system of generators (in the sense of Khabbaz). It is also shown that the only torsion free abelian groups which have q -covers are the free groups and that the torsion subgroup of an abelian group has a q -cover if the group does. These two facts give rise to the suspicion that there is a direct connection between groups with q -covers and splitting groups; two examples are given which show that this suspicion is false. Finally it is shown that very many abelian groups have i -covers; included among these are all torsion free groups and all groups which have q -covers.

Throughout this note all groups are abelian and the notation, for the most part, follows [1]. In particular: $r(G)$ is the rank of G , $r^*(G)$ is the reduced rank of G , G_t is the torsion subgroup of G , Z is the additive group of integers and Z_n is the integers mod n . A subgroup H of G is small in G if for any subgroup K of G , $\{H, K\} = G$ implies $K = G$.

DEFINITION (KHABBAZ [3]). A subset S of a group G is a minimum system of generators (abbreviated: M.s.g.) for G if $\{S\} = G$ and no finite subset S_1 of S may be replaced by a smaller subset T of G in such a way that $\{(S - S_1), T\} = G$.

DEFINITION. A q -cover for a group G is a free group F together with an epimorphism ϕ of F onto G such that for any proper direct summand F_1 of F , $\phi[F_1] \neq G$.

An i^* -cover of G is a free group F together with an epimorphism ϕ of F onto G such that if F_1 is free, ψ an epimorphism of F_1 onto G and σ is a homomorphism of F into F_1 such that $\psi\sigma = \phi$ then σ is a monomorphism.

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THEOREM 1. *The following are equivalent:*

- (I) *G has a q-cover.*
- (II) *G has a generating set S such that any finite subset of S is a M.s.g. for the subgroup it generates.*
- (III) *There is a free group F and an epimorphism ϕ of F onto G such that $\ker \phi$ contains no nonzero direct summand of F.*
- (IV) *G has an i^* -cover.*
- (V) *G has a M.s.g.*
- (VI) *There is a free group F and an epimorphism ϕ of F onto G such that $\ker \phi$ contains no nonzero pure subgroup of F.*

PROOF. (I) implies (II). Let $\phi: F \rightarrow G$ be a q -cover, $[z_i | i \in I]$ a basis of F and $S = [\phi(z_i) | i \in I]$. If $S_1 = [\phi(z_i) | i \in I']$ with $I' \subset I$ and I' finite and if $\{S_1\} = \{c_1, \dots, c_m\}$ with $m \leq |I'|$, then let $F_1 = \{z_i | i \in I'\}$ so that $\phi[F_1] = \{S_1\}$. Let H be a free group of rank m and σ a homomorphism of H onto $\{S_1\}$ which takes some basis of H onto $[c_1, \dots, c_m]$. If ρ is a homomorphism of F_1 into H such that $\rho\sigma = \phi|_{F_1}$, then $\ker \rho = 0$ since $\ker \rho$ is a direct summand of F_1 and hence of F . Hence $|I'| \leq m$ so $m = |I'|$.

(II) implies (III). Let S be as in (II), F a free group with basis $[z_s | s \in S]$ and let $\phi: F \rightarrow G$ be defined by $\phi(z_s) = s$. If $c = \sum r_s z_s$ is an element of $\ker \phi$ such that $F = \{c\} + H$, let $S_1 \subset S$ be finite such that $r_s = 0$ if $s \in S - S_1$ and let $F_1 = \{z_s | s \in S_1\}$. Thus $F_1 = \{c\} + F_1 \cap H$. However, $\phi[F_1] = \phi[F_1 \cap H]$ and $r(F_1 \cap H) + 1 = r(F_1) = |S_1|$, that is, $\{S_1\}$ may be generated by fewer than $|S_1|$ elements of G and (II) is contradicted.

(III) implies (IV). Let $\phi: F \rightarrow G$ be as in (III). If $\psi: F_1 \rightarrow G$ is an epimorphism with F_1 free and if $\sigma: F \rightarrow F_1$ such that $\psi\sigma = \phi$ then $\ker \sigma$ is a direct summand of F and $\ker \sigma$ is contained in $\ker \phi$ so by (III) $\ker \sigma = 0$ and σ is a monomorphism.

(IV) implies (I). Let $\phi: F \rightarrow G$ be an i^* -cover of G . If $F = F_2 + H$ and $\phi[F_1] = G$ then $\{F_1, \ker \phi\} = F$ and $\ker \phi = \ker \phi \cap F_1 + L$ for some L . Hence $F = \{F_1, \ker \phi \cap F_1 + L\} = F_1 + L$. Let $\psi = \phi|_{F_1}$ and let σ be the projection of $F_1 + L$ onto F_1 , then $\psi\sigma = \phi$ so σ is a monomorphism and $L = 0$. It follows that $H = 0$.

(II) is equivalent to (V). Let S be as in (II) and suppose it is not a M.s.g. Then there is a subset S_1 of S , S_1 is finite, and a subset T of G with $|T| < |S_1|$ such that $G = \{(S - S_1), T\}$. Choose $S_2 \subset S - (S_1 \cup T)$ such that S_2 is finite and $S_1 \subset \{T, S_2\}$; choose $S_3 \subset S - (S_1 \cup S_2)$ such that S_3 is finite and $T \subset \{S_1, S_2, S_3\}$. Since $S_1 \cup S_2 \subset \{T, S_2\}$ it follows that $\{T, S_2, S_3\} = \{S_1, S_2, S_3\}$. However, $|T \cup S_2 \cup S_3| \leq |T| + |S_2| + |S_3| < |S_1| + |S_2| + |S_3| = |S_1 \cup S_2 \cup S_3|$. That is, $S_1 \cup S_2 \cup S_3$ is

not a M.s.g. for $\{S_1, S_2, S_3\}$. This contradicts (II), hence (II) implies (V). It is trivial that (V) implies (II).

(III) is equivalent to (VI). This follows from the fact that any pure subgroup of a free group contains a direct summand of the same rank [1, p. 194] and conversely any direct summand is pure.

This completes the proof of Theorem 1. Notice that it has been shown that (II) can serve as a definition of M.s.g. Also, any i^* -cover is a q -cover and conversely. Finally if S is a M.s.g. for G then any free group with rank equal to $|S|$ together with any of the obvious epimorphisms forms a q -cover, and, conversely, if $\phi: F \rightarrow G$ is a q -cover then the image of any basis of F is a M.s.g. for G .

COROLLARY 1. *If G is torsion free then G has a M.s.g. iff G is free.*

COROLLARY 2. *If S and T are M.s.g.'s of G then $|S| = |T|$.*

COROLLARY 3. *If $r^*(G)$ is finite then G has a M.s.g. iff G is finitely generated.*

PROOF. If $r^*(G) = n$ then any finitely generated subgroup can be generated by fewer than $n + 1$ elements, hence if S is a M.s.g. $|S| \leq n$.

COROLLARY 4. *If G has a M.s.g. then so does G_t .*

PROOF. If $\phi: F \rightarrow G$ is a q -cover, $\phi^{-1}[G_t]$ is pure in F .

COROLLARY 5. *If G has a M.s.g. and G_t is small in G then G is free and hence $G_t = 0$.*

PROOF. If $\phi: F \rightarrow G$ is a q -cover and $\eta: G \rightarrow G/G_t$ is the canonical epimorphism then $\eta\phi: F \rightarrow G/G_t$ is a q -cover of G/G_t . Hence G/G_t is free and G_t is a direct summand and $G_t = 0$.

COROLLARY 6. *If G/G_t has a M.s.g. then G splits (that is G_t is a direct summand).*

COROLLARY 7. *If $r^*(G)$ is finite and G has a M.s.g. then G splits.*

In view of the last two corollaries it might be conjectured that if G has a M.s.g. then G splits. The following example shows that this is not the case.

EXAMPLE 1. Let $T = \sum_{i=1}^{\infty} \{a_i\}$ where the order of a_i is p^{2i} (for a fixed prime p). Let $U = \prod_{i=1}^{\infty} \{a_i\}$ and let $b_i \in U$ be given by

$$b_i = (0, 0, \dots, 0, a_i, pa_{i+1}, p^2a_{i+2}, \dots).$$

If $H = \{T, [b_i | i = 1, 2, \dots]\}$ then it is well known [1, p. 187] that H does not split. However, $[b_i | i = 1, 2, \dots]$ is a M.s.g. for H . It

generates since $a_i = b_i - pb_{i+1}$. It is minimum since $\{b_1, \dots, b_{n+1}\} = \{a_1\} + \dots + \{a_n\} + \{b_{n+1}\}$ so $\{b_1, \dots, b_{n+1}\}$ cannot be generated by fewer than $n+1$ elements or else its reduced rank would be less than $n+1$ which is not the case [1, p. 52].

The next example shows that the converse of Corollary 6 is false even if it is assumed in addition that G has a M.s.g.

EXAMPLE 2. Let (p_0, p_1, \dots) be the set of primes, let

$$Q' = \{1/p_i \mid i = 0, 1, \dots\}$$

and let K be a direct sum of a countable number of cyclic groups of order p_0 with e_i generating the i th group ($i=1, 2, \dots$). Let $G = Q' \times K$ and

$$S = [(1/p_i, e_i) \mid i = 1, 2, \dots] \cup [(1/p_0, 0)].$$

Since each pair $(1/p_i, 0)$ is in $\{S\}$ it follows that $Q' \subset \{S\}$ and hence $K \subset \{S\}$ so S generates G .

Let

$$S' = [(1/p_i, e_i) \mid i = 1, \dots, n] \cup [(1/p_0, 0)]$$

and let $H = \{S'\}$. Now, $Z \times \{0\} \subset H$ and S' is independent modulo $Z \times \{0\}$. Each pair $(1/p_i, e_i)$ has order $p_i p_0$ modulo $Z \times \{0\}$ hence $r^*(H/(Z \times \{0\})) \geq n+1$. It follows that $H/(Z \times \{0\})$ can be generated by no fewer than $n+1$ elements and hence neither can H . So S' is a M.s.g. for H . It follows immediately that S is a M.s.g. for G .

Finally G splits and $G/G_i \simeq Q'$ with $r^*(Q') = 1$ but Q' is not finitely generated and hence G/G_i has no M.s.g.

According to Corollary 3, if G is not finitely generated but has a M.s.g. then $r^*(G)$ is infinite. On the other hand if G_i is a direct sum of i copies of Z_{p_i} (p_i as in Example 2) and $G = \sum_{i=1}^{\infty} G_i$ then G is not finitely generated and $r^*(G)$ is infinite but, by [3, Theorem 6], G has no M.s.g.

If $\phi: F \rightarrow G$ and $\psi: F_1 \rightarrow G$ are q -covers, then by (IV) F and F_1 have the same rank and are thus isomorphic. However it is easy to construct an example such that there is no isomorphism ρ such that $\psi\rho = \phi$.

Now if $\phi: G \rightarrow D$ is a monomorphism with D divisible then D is a divisible hull of G iff for any monomorphism $\psi: G \rightarrow D_1$, with D_1 divisible, there is an epimorphism $\sigma: D_1 \rightarrow D$ such that $\sigma\psi = \phi$. The following definition is a dual of this statement.

DEFINITION. An i -cover of G is a free group F together with an epimorphism ϕ of F onto G such that, for any epimorphism ψ of F_1 onto G , F_1 free, there is a monomorphism σ of F into F_1 such that $\psi\sigma = \phi$.

Clearly any q -cover is an i -cover, however the following two theorems show that the converse is false.

THEOREM 2. *Any torsion free group has an i -cover.*

PROOF. If G is free there is nothing to show so suppose G is not free and in particular is not finitely generated.

Choose an epimorphism $\phi: F \rightarrow G$ with F free and $r(\ker \phi)$ minimal among all such epimorphisms. It will be shown that $\phi: F \rightarrow G$ is an i -cover, so suppose H is free and $\psi: H \rightarrow G$ is an epimorphism. Let $\sigma: F \rightarrow H$ be such that $\psi\sigma = \phi$ and it may as well be assumed that $\sigma[F] = H$.

There is a direct summand K of F such that $\ker \phi \subset K$ and $r(K) = r(\ker \phi)$. If $F = K + F_1$ then F/K is isomorphic to $G/\phi[K]$ by the map induced by ϕ . Since F/K is free, $G = \phi[K] + G_1$ for some G_1 and $\phi|_{F_1}$ is an isomorphism of F_1 onto G_1 . Now $\ker \sigma \subset K$ and $\ker \psi \subset \sigma[K]$ so that $H = \sigma[K] + \sigma[F_1]$ and $\psi|_{\sigma[F_1]}$ and $\sigma|_{F_1}$ are isomorphisms onto G_1 and $\sigma[F_1]$ respectively. The problem reduces to that of finding a commuting monomorphism of K into $\sigma[K]$.

Now $r(\ker \psi) \leq r(\sigma[K]) \leq r(K) = r(\ker \phi)$ so from the minimality of $r(\ker \phi)$ it follows that $r(\ker \psi) = r(\ker \phi)$.

If $K/\ker \phi$ is not of finite rank $|K/\ker \phi| = r(K/\ker \phi)$ and there is a free group F' with $r(F') = r(K/\ker \phi)$ and an epimorphism ϕ' of $F' + F_1$ onto G with $\ker \phi' \subset F'$ so, since $r(K/\ker \phi) \leq r(K)$, it follows that $r(K/\ker \phi) = r(K)$ or else $r(\ker \phi') < r(\ker \phi)$, contradicting the minimality of $r(\ker \phi)$. Since $\ker \phi$ is pure in K , $\ker \phi$ contains a direct summand S of K such that $r(S) = r(\ker \phi)$. So, in any case, $\ker \phi$ contains a summand S of K such that $r(S) = |K/\ker \phi|$. Similarly $\ker \psi$ contains a direct summand S' of $\sigma[K]$ such that $r(S') = |\sigma[K]/\ker \psi|$.

Now $K/\ker \phi$ is isomorphic to $\sigma[K]/\ker \psi$ by the map induced by ϕ and ψ , K and $\sigma[K]$ have infinite rank since $\ker \phi$ and $\ker \psi$ have infinite rank so $r(K) = r(\sigma[K])$, $r(S) = |K/\ker \phi|$ and $r(S') = |\sigma[K]/\ker \psi|$. It follows from a theorem of Erdős [1, p. 195] that there is an isomorphism θ of K onto $\sigma[K]$ such that θ commutes with ϕ and ψ on K as desired.

THEOREM 3. *If G is not finitely generated and G has the property that every epimorphism ϕ of a free group F onto G with $r(F) = |G|$ is such that $\ker \phi$ contains a direct summand of F with rank equal to $r(F)$, then G has an i -cover.*

PROOF. Let F be free with $r(F) = |G|$ and let $\phi: F \rightarrow G$ be an epimorphism. It will be shown that this is an i -cover.

Let H be free and ψ an epimorphism of H onto G . It may be assumed that $r(H) = |G|$. According to the hypothesis both $\ker \phi$ and $\ker \psi$ contain direct summands S and S_1 of F and H respectively such that $r(S) = r(S_1) = r(F) = r(H) = |G|$. Moreover, since $r(S) = |F/\ker \phi|$ and $r(S_1) = |H/\ker \psi|$ and $F/\ker \phi$ is isomorphic to $H/\ker \psi$ by the map induced by ϕ and ψ , the theorem of Erdős cited above applies and F is isomorphic to H by a commuting isomorphism.

COROLLARY 8. *Any countable group has an i -cover.*

PROOF. If G has a q -cover then G has an i -cover; if G is countable and has no q -cover then Theorem 3 applies.

REMARK. Among those groups with i -covers are those which have q -covers and those which very badly miss having q -covers. It should also be noted that in Theorem 2 and Theorem 3 an i -cover was very easy to get: all that was needed was a free group of the proper rank and an epimorphism. One might hope G would have a quasi-projective cover (cf. [2]) iff G has a q -cover. This is wishful thinking however since among the finitely generated, nontorsion free groups only the finite groups have quasi-projective covers.

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