ON GENERATING SYSTEMS FOR ABELIAN GROUPS

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The purpose of this note is to investigate two kinds of generating sets for an abelian group. It is well known that if an abelian group has a projective cover then it has a very special kind of generating set: a free basis. The idea of a projective cover of an abelian group is dual to that of a divisible hull when the divisible hull is defined as an essential injective extension. This note considers the dualization of two other equivalent definitions of a divisible hull. This yields the definitions of a q-cover and an i-cover. It is shown that an abelian group has a q-cover if and only if it has a minimum system of generators (in the sense of Khabbaz). It is also shown that the only torsion free abelian groups which have q-covers are the free groups and that the torsion subgroup of an abelian group has a q-cover if the group does. These two facts give rise to the suspicion that there is a direct connection between groups with q-covers and splitting groups; two examples are given which show that this suspicion is false. Finally it is shown that very many abelian groups have i-covers; included among these are all torsion free groups and all groups which have q-covers.

Throughout this note all groups are abelian and the notation, for the most part, follows [1]. In particular: $r(G)$ is the rank of G, $r^*(G)$ is the reduced rank of G, G_t is the torsion subgroup of G, Z is the additive group of integers and Z_n is the integers mod n. A subgroup H of G is small in G if for any subgroup K of G, $\{H, K\}=G$ implies $K=G$.

Definition (Khabbaz [3]). A subset S of a group G is a minimum system of generators (abbreviated: M.s.g.) for G if $\{S\}=G$ and no finite subset S_1 of S may be replaced by a smaller subset T of G in such a way that $\{(S-S_1), T\}=G$.

Definition. A q-cover for a group G is a free group F together with an epimorphism ϕ of F onto G such that for any proper direct summand F_1 of F, $\phi[F_1]\neq G$.

An i^*-cover of G is a free group F together with an epimorphism ϕ of F onto G such that if F_1 is free, ψ an epimorphism of F_1 onto G and σ is a homomorphism of F into F_1 such that $\psi\sigma=\phi$ then σ is a monomorphism.

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THEOREM 1. *The following are equivalent*:

(I) *G has a q-cover.*

(II) *G has a generating set S such that any finite subset of S is a M.s.g. for the subgroup it generates.*

(III) *There is a free group F and an epimorphism ϕ of F onto G such that* ker ϕ *contains no nonzero direct summand of F.*

(IV) *G has an i^*-cover.*

(V) *G has a M.s.g.*

(VI) *There is a free group F and an epimorphism ϕ of F onto G such that* ker ϕ *contains no nonzero pure subgroup of F.*

PROOF. (I) implies (II). Let ϕ: $F \rightarrow G$ be a q-cover, $[z_i | i \in I]$ a basis of F and $S = [\phi(z_i) | i \in I]$. If $S_1 = [\phi(z_i) | i \in I']$ with $I' \subset I$ and I' finite and if $\{S_1\} = \{c_1, \cdots, c_m\}$ with $m \leqq |I'|$, then let $F_1 = \{z_i | i \in I'\}$ so that $\phi[F_1] = \{S_1\}$. Let H be a free group of rank m and σ a homomorphism of H onto $\{S_1\}$ which takes some basis of H onto $[c_1, \cdots, c_m]$. If ρ is a homomorphism of F_1 into H such that $\rho\sigma = \phi | F_1$, then ker $\rho = 0$ since ker ρ is a direct summand of F_1 and hence of F. Hence $|I'| \leqq m$ so $m = |I'|$.

(II) implies (III). Let S be as in (II), F a free group with basis $[z_s | s \in S]$ and let ϕ: $F \rightarrow G$ be defined by $\phi(z_s) = s$. If $c = \sum r_s z_s$ is an element of ker ϕ such that $F = \{c\} + H$, let $S_1 \subset S$ be finite such that $r_s = 0$ if $s \in S - S_1$ and let $F_1 = \{z_s | s \in S_1\}$. Thus $F_1 = \{c\} + F_1 \cap H$. However, $\phi[F_1] = \phi[F_1 \cap H]$ and $r(F_1 \cap H) + 1 = r(F_1) = |S_1|$, that is, $\{S_1\}$ may be generated by fewer than $|S_1|$ elements of G and (II) is contradicted.

(III) implies (IV). Let ϕ: $F \rightarrow G$ be as in (III). If ψ: $F_1 \rightarrow G$ is an epimorphism with F_1 free and if σ: $F \rightarrow F_1$ such that $\psi\sigma = \phi$ then ker σ is a direct summand of F and ker σ is contained in ker ϕ so by (III) ker $\sigma = 0$ and σ is a monomorphism.

(IV) implies (I). Let ϕ: $F \rightarrow G$ be an i^*-cover of G. If $F = F_2 + H$ and $\phi[F_1] = G$ then $\{F_1, \ker \phi\} = F$ and ker $\phi = \ker \phi \cap F_1 + L$ for some L. Hence $F = \{F_1, \ker \phi \cap F_1 + L\} = F_1 + L$. Let $\psi = \phi | F_1$ and let σ be the projection of $F_1 + L$ onto F_1, then $\psi\sigma = \phi$ so σ is a monomorphism and $L = 0$. It follows that $H = 0$.

(II) is equivalent to (V). Let S be as in (II) and suppose it is not a M.s.g. Then there is a subset S_1 of S, S_1 is finite, and a subset T of G with $|T| < |S_1|$ such that $G = \{(S - S_1), T\}$. Choose $S_2 \subset S - (S_1 \cup T)$ such that S_2 is finite and $S_1 \subset \{T, S_2\}$; choose $S_3 \subset S - (S_1 \cup S_2)$ such that S_3 is finite and $T \subset \{S_1, S_2, S_3\}$. Since $S_1 \cup S_2 \subset \{T, S_2\}$ it follows that $\{T, S_2, S_3\} = \{S_1, S_2, S_3\}$. However, $|T \cup S_2 \cup S_3| \leqq |T| + |S_2| + |S_3| < |S_1| + |S_2| + |S_3| = |S_1 \cup S_2 \cup S_3|$. That is, $S_1 \cup S_2 \cup S_3$ is not a M.s.g. for $\{S_1, S_2, S_3\}$. This contradicts (II), hence (II) implies (V). It is trivial that (V) implies (II).

(III) is equivalent to (VI). This follows from the fact that any pure subgroup of a free group contains a direct summand of the same rank [1, p. 194] and conversely any direct summand is pure.

This completes the proof of Theorem 1. Notice that it has been shown that (II) can serve as a definition of M.s.g. Also, any i^*-cover is a q-cover and conversely. Finally if S is a M.s.g. for G then any free group with rank equal to $|S|$ together with any of the obvious epimorphisms forms a q-cover, and, conversely, if $\phi: F \to G$ is a q-cover then the image of any basis of F is a M.s.g. for G.

COROLLARY 1. *If G is torsion free then G has a M.s.g. iff G is free.*

COROLLARY 2. *If S and T are M.s.g.'s of G then $|S| = |T|$.*

COROLLARY 3. *If $r^*(G)$ is finite then G has a M.s.g. iff G is finitely generated.*

PROOF. If $r^*(G)=n$ then any finitely generated subgroup can be generated by fewer then $n+1$ elements, hence if S is a M.s.g. $|S| \leqq n$.

COROLLARY 4. *If G has a M.s.g. then so does G_t.*

PROOF. If $\phi: F \to G$ is a q-cover, $\phi^{-1}[G_t]$ is pure in F.

COROLLARY 5. *If G has a M.s.g. and G_t is small in G then G is free and hence $G_t = 0$.*

PROOF. If $\phi: F \to G$ is a q-cover and $\eta: G \to G/G_t$ is the canonical epimorphism then $\eta\phi: F \to G/G_t$ is a q-cover of G/G_t. Hence G/G_t is free and G_t is a direct summand and $G_t = 0$.

COROLLARY 6. *If G/G_t has a M.s.g. then G splits (that is G_t is a direct summand).*

COROLLARY 7. *If $r^*(G)$ is finite and G has a M.s.g. then G splits.*

In view of the last two corollaries it might be conjectured that if G has a M.s.g. then G splits. The following example shows that this is not the case.

EXAMPLE 1. Let $T = \sum_{i=1}^{\infty} \{a_i\}$ where the order of a_i is p^{2i} (for a fixed prime p). Let $U = \prod_{i=1}^{\infty} \{a_i\}$ and let $b_i \in U$ be given by

$$b_i = (0, 0, \cdots, 0, a_i, pa_{i+1}, p^2a_{i+2}, \cdots).$$

If $H = \{T, [b_i | i=1, 2, \cdots]\}$ then it is well known [1, p. 187] that H does not split. However, $[b_i | i=1, 2, \cdots]$ is a M.s.g. for H. It generates since $a_i = b_i - pb_{i+1}$. It is minimum since $\{b_1, \cdots, b_{n+1}\} = \{a_1\} + \cdots + \{a_n\} + \{b_{n+1}\}$ so $\{b_1, \cdots, b_{n+1}\}$ cannot be generated by fewer than $n+1$ elements or else its reduced rank would be less than $n+1$ which is not the case [**1**, p. 52].

The next example shows that the converse of Corollary 6 is false even if it is assumed in addition that G has a M.s.g.

EXAMPLE 2. Let $(p_0, p_1, \cdots)$ be the set of primes, let

$$Q' = \{1/p_i \mid i = 0, 1, \cdots\}$$

and let K be a direct sum of a countable number of cyclic groups of order p_0 with e_i generating the ith group $(i = 1, 2, \cdots)$. Let $G = Q' \times K$ and

$$S = [(1/p_i, e_i) \mid i = 1, 2, \cdots] \cup [(1/p_0, 0)].$$

Since each pair $(1/p_i, 0)$ is in $\{S\}$ it follows that $Q' \subset \{S\}$ and hence $K \subset \{S\}$ so S generates G.

Let

$$S' = [(1/p_i, e_i) \mid i = 1, \cdots, n] \cup [(1/p_0, 0)]$$

and let $H = \{S'\}$. Now, $Z \times \{0\} \subset H$ and S' is independent modulo $Z \times \{0\}$. Each pair $(1/p_i, e_i)$ has order $p_i p_0$ modulo $Z \times \{0\}$ hence $r^*(H/(Z \times \{0\})) \geqq n+1$. It follows that $H/(Z \times \{0\})$ can be generated by no fewer than $n+1$ elements and hence neither can H. So S' is a M.s.g. for H. It follows immediately that S is a M.s.g. for G.

Finally G splits and $G/G_t \simeq Q'$ with $r^*(Q') = 1$ but Q' is not finitely generated and hence G/G_t has no M.s.g.

According to Corollary 3, if G is not finitely generated but has a M.s.g. then $r^*(G)$ is infinite. On the other hand if G_i is a direct sum of i copies of Z_{p_i} (p_i as in Example 2) and $G = \sum_{i=1}^{\infty} G_i$ then G is not finitely generated and $r^*(G)$ is infinite but, by [**3**, Theorem 6], G has no M.s.g.

If $\phi: F \to G$ and $\psi: F_1 \to G$ are q-covers, then by (IV) F and F_1 have the same rank and are thus isomorphic. However it is easy to construct an example such that there is no isomorphism ρ such that $\psi\rho = \phi$.

Now if $\phi: G \to D$ is a monomorphism with D divisible then D is a divisible hull of G iff for any monomorphism $\psi: G \to D_1$, with D_1 divisible, there is an epimorphism $\sigma: D_1 \to D$ such that $\sigma\psi = \phi$. The following definition is a dual of this statement.

DEFINITION. An i-cover of G is a free group F together with an epimorphism ϕ of F onto G such that, for any epimorphism ψ of F_1 onto G, F_1 free, there is a monomorphism σ of F into F_1 such that $\psi\sigma = \phi$.

Clearly any q-cover is an i-cover, however the following two theorems show that the converse is false.

THEOREM 2. *Any torsion free group has an i-cover.*

PROOF. If G is free there is nothing to show so suppose G is not free and in particular is not finitely generated.

Choose an epimorphism $\phi: F\to G$ with F free and $r(\ker \phi)$ minimal among all such epimorphisms. It will be shown that $\phi: F\to G$ is an i-cover, so suppose H is free and $\psi: H\to G$ is an epimorphism. Let $\sigma: F\to H$ be such that $\psi\sigma=\phi$ and it may as well be assumed that $\sigma[F]=H$.

There is a direct summand K of F such that $\ker \phi\subset K$ and $r(K)=r(\ker \phi)$. If $F=K+F_1$ then F/K is isomorphic to $G/\phi[K]$ by the map induced by ϕ. Since F/K is free, $G=\phi[K]+G_1$ for some G_1 and $\phi| F_1$ is an isomorphism of F_1 onto G_1. Now $\ker \sigma\subset K$ and $\ker \psi\subset\sigma[K]$ so that $H=\sigma[K]+\sigma[F_1]$ and $\psi|\sigma[F_1]$ and $\sigma| F_1$ are isomorphisms onto G_1 and $\sigma[F_1]$ respectively. The problem reduces to that of finding a commuting monomorphism of K into $\sigma[K]$.

Now $r(\ker \psi)\leqq r(\sigma[K])\leqq r(K)=r(\ker \phi)$ so from the minimality of $r(\ker \phi)$ it follows that $r(\ker \psi)=r(\ker \phi)$.

If $K/\ker \phi$ is not of finite rank $|K/\ker \phi|=r(K/\ker \phi)$ and there is a free group F' with $r(F')=r(K/\ker \phi)$ and an epimorphism ϕ' of $F'+F_1$ onto G with $\ker \phi'\subset F'$ so, since $r(K/\ker \phi)\leqq r(K)$, it follows that $r(K/\ker \phi)=r(K)$ or else $r(\ker \phi')<r(\ker \phi)$, contradicting the minimality of $r(\ker \phi)$. Since $\ker \phi$ is pure in K, $\ker \phi$ contains a direct summand S of K such that $r(S)=r(\ker \phi)$. So, in any case, $\ker \phi$ contains a summand S of K such that $r(S)=|K/\ker \phi|$. Similarly $\ker \psi$ contains a direct summand S' of $\sigma[K]$ such that $r(S')=|\sigma[K]/\ker \psi|$.

Now $K/\ker \phi$ is isomorphic to $\sigma[K]/\ker \psi$ by the map induced by ϕ and ψ, K and $\sigma[K]$ have infinite rank since $\ker \phi$ and $\ker \psi$ have infinite rank so $r(K)=r(\sigma[K])$, $r(S)=|K/\ker \phi|$ and $r(S')=|\sigma[K]/\ker \psi|$. It follows from a theorem of Erdös [**1**, p. 195] that there is an isomorphism θ of K onto $\sigma[K]$ such that θ commutes with ϕ and ψ on K as desired.

THEOREM 3. *If G is not finitely generated and G has the property that every epimorphism ϕ of a free group F onto G with $r(F)=|G|$ is such that* $\ker \phi$ *contains a direct summand of F with rank equal to $r(F)$, then G has an i-cover.*

PROOF. Let F be free with $r(F)=|G|$ and let $\phi: F\to G$ be an epimorphism. It will be shown that this is an i-cover.

Let H be free and ψ an epimorphism of H onto G. It may be assumed that $r(H) = |G|$. According to the hypothesis both ker ϕ and ker ψ contain direct summands S and S_1 of F and H respectively such that $r(S) = r(S_1) = r(F) = r(H) = |G|$. Moreover, since $r(S) = |F/\ker \phi|$ and $r(S_1) = |H/\ker \psi|$ and $F/\ker \phi$ is isomorphic to $H/\ker \psi$ by the map induced by ϕ and ψ, the theorem of Erdös cited above applies and F is isomorphic to H by a commuting isomorphism.

COROLLARY 8. *Any countable group has an i-cover.*

PROOF. If G has a q-cover then G has an i-cover; if G is countable and has no q-cover then Theorem 3 applies.

REMARK. Among those groups with i-covers are those which have q-covers and those which very badly miss having q-covers. It should also be noted that in Theorem 2 and Theorem 3 an i-cover was very easy to get: all that was needed was a free group of the proper rank and an epimorphism. One might hope G would have a quasi-projective cover (cf. [2]) iff G has a q-cover. This is wishful thinking however since among the finitely generated, nontorsion free groups only the finite groups have quasi-projective covers.

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