INFINITELY MANY NONISOMORPHIC NILPOTENT ALGEBRAS

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1. Introduction. In [**1**], the following theorem was proved:

THEOREM A. *There exist uncountably many nonisomorphic nilpotent Lie algebras of class* 3 ($L\supset L^2=[L, L]\supset 0$) *over the real number field for any given dimension* $\geqq 10$.

The main purpose here is in §2, by using a similar method as in [**2**, p. 72], to prove a generalization of Theorem A. Namely, the following

THEOREM 1. *Let K be a field whose characteristic is not* 2 *and whose cardinality is* $\aleph_0$ (*greater than* $\aleph_0$). *Then there exist countably infinitely* (*uncountably*) *many nonisomorphic nilpotent Lie algebras of class* 3 *over K for any given dimension* $\geqq 10$.

Following from Theorem 1 we can easily show

THEOREM 2. *Let K be a field whose characteristic is not* 2 *and whose cardinality is* $\aleph_0$ (*greater than* $\aleph_0$). *Then there exist countably infinitely* (*uncountably*) *many nonisomorphic solvable not nilpotent Lie algebras of index* 3 ($L\supset L^2\supset [L^2, L^2]\supset 0$) *over K for any given dimension* $\geqq 11$.

In §3, we point out that the method used in [**1**] can also be applied to nilpotent associative algebras.

2. **Nilpotent Lie algebras.** Let L be a Lie algebra of dimension $m+n$ over a field F defined by a basis $(x_1, \cdots, x_m, y_1, \cdots, y_n)$ with a bilinear antisymmetric bracket multiplication such that

$$(1) \qquad [x_j, x_k] = \sum_{i=1}^{n} c_{jk}^{i} y_i, \qquad j, k = 1, 2, \cdots, m$$

where $c_{jk}^i \in F$ and not all c_{jk}^i are zero and all other products are zero. Then L is a nilpotent Lie algebra of class 3. The constants c_{jk}^i are called the structure constants, and the matrices $C^{(i)}=(c_{jk}^i)$ are called the structure matrices. We assume that the center of L, $Z(L)$, is the commutator $L^2=[L, L]$ of L and we determine how the structure matrices change as we shift to another basis $(z_1, \cdots, z_m, z_{m+1}, \cdots,$

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z_{m+n}) of L with structure constants d^i_{jk} in F, $1 \leqq i, j, k \leqq m+n$. We may assume that $(z_1, z_2, \cdots, z_m)$ are independent modulo L^2, i.e., they span a complement V of L^2 in L. We can write $z_{m+i} = v_i + t_i$ with $v_i \in V$ and $t_i \in L^2$ for $i = 1, 2, \cdots, n$. Clearly, $(z_1, \cdots, z_m, t_1, \cdots, t_n)$ is still a basis for L. Since $[z_i, z_j] \in L^2$, we have

$$[z_i, z_j] = \sum_{u=1}^{n} d_{ij}^{m+u} t_u, \qquad i, j = 1, 2, \cdots, m \tag{2}$$

and all other products are zero. We note that $(x_1, x_2, \cdots, x_m)$ also forms a complement T of L^2 in L. Hence, we can replace each z_i by an element s_i such that $s_i - z_i \in L^2$ and $s_i \in T$. Since L^2 is the center of L, the structure constants for the basis $(s_1, \cdots, s_m, t_1, \cdots, t_n)$ are the same as for the basis $(z_1, \cdots, z_m, t_1, \cdots, t_n)$ above.

The set of vectors $(s_1, \cdots, s_m)$ is a basis for T. Hence, we have $s_i = \sum_{p=1}^{m} a_{ip} x_p$, $i = 1, 2, \cdots, m$ where $A = (a_{ip})$ is a nonsingular matrix. Similarly, $y_r = \sum_{u=1}^{n} b_{ru} t_u$, $i = 1, 2, \cdots, n$, where $B = (b_{ru})$ is a nonsingular matrix. Then on the one hand $[s_i, s_j] = \sum_{u=1}^{n} d_{ij}^{m+u} t_u$, and on the other hand

$$[s_i, s_j] = \left[\sum_{p=1}^{m} a_{ip} x_p, \ \sum_{v=1}^{m} a_{jv} x_v \right]$$

$$= \sum_{p} \sum_{v} a_{ip} a_{jv} \sum_{r} c_{pv}^{r} \sum_{u} b_{ru} t_u.$$

By the linear independence of t_u's, we have

$$d_{ij}^{m+u} = \sum_{p} \sum_{v} \sum_{r} a_{ip} a_{jv} c_{pv}^{r} b_{ru}.$$

In the form of matrices, we have

$$D^{(u)} = \sum_{r=1}^{u} b_{ru} A C^{(r)} A', \qquad u = 1, 2, \cdots, n \tag{3}$$

where $D^{(u)}$ are the structure matrices of (d_{ij}^{m+u}) and A' is the transpose matrix of A. In particular, we have

LEMMA 1. *Let L be a nilpotent Lie algebra of class* 3 *with dimension $m+n$ over a field F. The multiplication of L is defined by* (1) *such that $Z(L) = L^2$. If M is a Lie algebra isomorphic to L then* (3) *holds where $D^{(u)}$, $u = 1, 2, \cdots, n$, are structure matrices of M.*

PROOF. Let θ be the isomorphic map of M onto L. Then M is a nilpotent Lie algebra of class 3 with dimension $m+n$ over F such that

$Z(M)=M^2$. Say, $M=((z_1', \cdots, z_m', t_1', \cdots, t_n'))$ with

$$[z_i', z_j'] = \sum_{u=1}^{n} d_{ij}^{m+u} t_u', \qquad i, j = 1, 2, \cdots, m,$$

and all other products are zero. Let $z_i'\theta = z_i$ for $i=1, 2, \cdots, m$, and $t_u'\theta = t_u$ for $u=1, 2, \cdots, n$. Since θ is an isomorphism of Lie algebras, we obtain (2). Consequently, (3) follows.

Let K be an infinite field whose characteristic is not 2 and let $L_\alpha = ((x_1, x_2, \cdots, x_8, y_1, y_2))$ be an 10 dimensional Lie algebra over a field K defined by

$$[x_1, x_2] = y_1 + y_2, \qquad [x_3, x_4] = y_1 + 2y_2,$$
$$[x_5, x_6] = y_1, \qquad [x_7, x_8] = y_1 + \alpha y_2$$

where $\alpha \in K$, $\alpha \neq 0, 1, 2$ and all other products are zero. Clearly, L_α is a nilpotent Lie algebra of class 3. For convenience, we let J and C_α be the structure matrices $C^{(1)} = (c_{ij}^1)$ and $C^{(2)} = (c_{ij}^2)$ of L_α respectively, i.e.,

$$J = \begin{bmatrix} 0 & 1 & & & & & & \\ -1 & 0 & & & & & & \\ & & 0 & 1 & & & & \\ & & -1 & 0 & & & & \\ & & & & 0 & 1 & & \\ & & & & -1 & 0 & & \\ & & & & & & 0 & 1 \\ & & & & & & -1 & 0 \end{bmatrix} \quad \text{and} \quad C_\alpha = \begin{bmatrix} 0 & 1 & & & & & & \\ -1 & 0 & & & & & & \\ & & 0 & 2 & & & & \\ & & -2 & 0 & & & & \\ & & & & 0 & 0 & & \\ & & & & 0 & 0 & & \\ & & & & & & 0 & \alpha \\ & & & & & & -\alpha & 0 \end{bmatrix}.$$

Since $\alpha \neq 0, 1, 2$, any two different nonzero products are linearly independent, and L_α^2 is generated by y_1 and y_2. Let $x \in Z(L)$. Then $x = \sum_{i=1}^8 a_i x_i + \sum_{j=1}^2 b_j y_j$ and

$$0 = [x, x_k] = \sum_{i=1}^{8} a_i \sum_{t=1}^{2} c_{ik}^t y_t \qquad \text{for } k = 1, 2, \cdots, 8.$$

By the linear independence of y_t's, we have $\sum_{i=1}^8 a_i c_{ik}^t = 0$ for $t=1, 2$ and $k=1, 2, \cdots, 8$. By using $\sum_{i=1}^8 a_i c_{ik}^1 = 0$, $k=1, 2, \cdots, 8$, we have $a_1 = a_2 = \cdots = a_8 = 0$. Hence, $x = \sum_{j=1}^2 b_j y_j \in L^2$ and $Z(L) = L^2$.

LEMMA 2. *Let α be an element of an infinite field K whose characteristic is not 2 such that $\alpha(\alpha-1)(\alpha-2)(3\alpha-4)(3\alpha-2) \neq 0$. Then there are* *only finitely many β such that $L_\beta \cong L_\alpha$.*

PROOF. Suppose $L_\beta \cong L_\alpha$. Then by (3) we have

$$J = b_{11}AJA' + b_{21}AC_\alpha A',$$
$$C_\beta = b_{12}AJA' + b_{22}AC_\alpha A'.$$

Multiply both equations by $J^{-1} = -J$ on the left, we obtain:

$$I = J^{-1}J = b_{11}J^{-1}AJA' + b_{21}J^{-1}AC_\alpha A', \tag{4}$$

$$J^{-1}C_\beta = b_{12}J^{-1}AJA' + b_{22}J^{-1}AC_\alpha A'. \tag{5}$$

Since I and $J^{-1}C_\beta$ are diagonal matrices, so are $J^{-1}AJA'$ and $J^{-1}AC_\alpha A'$. Let $d = J^{-1}AJA'$. Then

$$d = \{d_1, d_1, d_2, d_2, d_3, d_3, d_4, d_4\} = \frac{1}{|B|}(b_{22}I - b_{21}J^{-1}C_\beta)$$

where $\{\quad\}$ denotes a diagonal matrix, $|B|$ is the determinant of B,

$$d_1 = \frac{1}{|B|}(b_{22} - b_{21}), \qquad d_2 = \frac{1}{|B|}(b_{22} - 2b_{21}),$$

$$d_3 = \frac{1}{|B|}b_{22} \qquad \text{and} \qquad d_4 = \frac{1}{|B|}(b_{22} - \beta b_{21}).$$

Since $A' = J^{-1}A^{-1}Jd$ and $J^{-1}AC_\alpha A'$ is a diagonal matrix, $J^{-1}AC_\alpha J^{-1}A^{-1}Jd$ is a diagonal matrix. Consequently, $J^{-1}AC_\alpha J^{-1}A^{-1}J = (J^{-1}A)(C_\alpha J^{-1})(J^{-1}A)^{-1}$ is a diagonal matrix whose eigenvalues are the same as the eigenvalues of $C_\alpha J^{-1}$, namely, 1, 1, 2, 2, 0, 0, α, α.

Let $\omega_1, \omega_2, \cdots, \omega_8$ be a permutation of 1, 1, 2, 2, 0, 0, α, α. Then (4) and (5) can be written as

$$I = b_{11}\{d_1, \cdots, d_4\} + b_{21}\{\omega_1, \cdots, \omega_8\}\{d_1, \cdots, d_4\}, \tag{6}$$

$$\begin{aligned}&\{1, 1, 2, 2, 0, 0, \beta, \beta\} \\ &\qquad = b_{12}\{d_1, \cdots, d_4\} + b_{22}\{\omega_1, \cdots, \omega_8\}\{d_1, \cdots, d_4\}.\end{aligned} \tag{7}$$

Since $d = J^{-1}AJA'$ and J and A are nonsingular, d is nonsingular. Hence, $b_{22} \neq 0$. Write the matrix equation (7) as 8 equations, and using $b_{22} \neq 0$, we have $\omega_1 = \omega_2$, $\omega_3 = \omega_4$, $\omega_5 = \omega_6$ and $\omega_7 = \omega_8$. Then we have 4 possible cases:

Case 1. $J^{-1}AC_\alpha J^{-1}A^{-1}J = \{0, 0. \omega_4, \omega_4, \omega_6, \omega_6, \omega_8, \omega_8\}$,
Case 2. $J^{-1}AC_\alpha J^{-1}A^{-1}J = \{\omega_4, \omega_4, 0, 0, \omega_6, \omega_6, \omega_8, \omega_8\}$,
Case 3. $J^{-1}AC_\alpha J^{-1}A^{-1}J = \{\omega_4, \omega_4, \omega_6, \omega_6, 0, 0, \omega_8, \omega_8\}$,
Case 4. $J^{-1}AC_\alpha J^{-1}A^{-1}J = \{\omega_4, \omega_4, \omega_6, \omega_6, \omega_8, \omega_8, 0, 0\}$

where $\omega_4, \omega_6, \omega_8$ is a permutation of 1, 2, α.

Now we consider the Case 1: The matrix equation (6) can be written as:

$$1 = b_{11}d_1, \tag{8}$$

$$1 = b_{11}d_2 + b_{21}\omega_4 d_2, \tag{9}$$

$$1 = b_{11}d_3 + b_{21}\omega_6 d_3, \tag{10}$$

$$1 = b_{11}d_4 + b_{21}\omega_8 d_4. \tag{11}$$

Also, the matrix equation (7) can be written as:

$$1 = b_{12}d_1, \tag{12}$$

$$2 = b_{12}d_2 + b_{22}\omega_4 d_2, \tag{13}$$

$$0 = b_{12}d_3 + b_{22}\omega_6 d_3, \tag{14}$$

$$\beta = b_{12}d_4 + b_{22}\omega_8 d_4. \tag{15}$$

We may assume $d_1 = 1$. The reason is that the equations (4) and (5) still hold if we replace A by (λA) and B by $(\lambda^{-2}B)$ where $\lambda \neq 0$ and λ belongs to K or λ belongs to an extension field of K. We have

$$I = b'_{11}J^{-1}EJE' + b'_{21}J^{-1}EC_\alpha E', \tag{4'}$$

$$J^{-1}C_\beta = b'_{12}J^{-1}EJE' + b'_{22}J^{-1}EC_\alpha E' \tag{5'}$$

where $b'_{ij} = b_{ij}\lambda^{-2}$, $i, j = 1, 2$ and $E = (\lambda A)$. We choose λ^2 to be $|B|/(b_{22} - b_{21})$, we have

$$J^{-1}EJE' = \frac{1}{|\overline{B}|}(b'_{22}I - b'_{21}J^{-1}C_\beta) = \{1, 1, \cdots\}$$

$$\text{where } |\overline{B}| = b'_{11}b'_{22} - b'_{12}b'_{21}.$$

Hence, we may assume $d_1 = 1$.

From (8) and (12), we have $d_1 = b_{11} = b_{12} = 1$. By (14), $b_{22} = -1/\omega_6$. By (13), we have

$$d_2 = 2\omega_6/(\omega_6 - \omega_4).$$

By (9), we have

$$b_{21} = -(\omega_6 + \omega_4)/2\omega_4\omega_6.$$

By (10), we have

$$d_3 = 2\omega_4/(\omega_4 - \omega_6).$$

By (11), we have

$$d_4 = 2\omega_4\omega_6/(2\omega_4\omega_6 - \omega_6\omega_8 - \omega_4\omega_8).$$

Substitute into (15), we obtain

$$\beta = 2\omega_4(\omega_6 - \omega_8)/(2\omega_4\omega_6 - \omega_6\omega_8 - \omega_4\omega_8).$$

Since α satisfies the condition in the lemma, $2\omega_4\omega_6-\omega_6\omega_8-\omega_4\omega_8\neq 0$.

In the following cases, we may still assume $d_1=1$. By using similar computations as in the Case 1, we obtain:

In Case 2, $\beta=2\omega_4(\omega_8-\omega_6)/(-\omega_4\omega_6+2\omega_4\omega_8-\omega_6\omega_8)$.

In Case 3, $\beta=2\omega_8(\omega_4-\omega_6)/(-\omega_4\omega_6+2\omega_4\omega_8-\omega_6\omega_8)$.

In Case 4, $\beta=2\omega_8(\omega_6-\omega_4)/(-\omega_4\omega_6-\omega_4\omega_8+2\omega_6\omega_8)$.

Hence, there are only finitely many β such that $L_\beta\cong L_\alpha$. In fact, for each α, the possible β's are:

$$\alpha, \quad -(\alpha - 2), \quad \frac{2\alpha}{3\alpha - 2}, \quad \frac{4 - 2\alpha}{4 - 3\alpha}, \quad \frac{4 - 4\alpha}{4 - 3\alpha} \quad \text{and} \quad \frac{4\alpha - 4}{3\alpha - 2}.$$

Now the proof of Theorem 1 goes as follows: Let K' be the set consisting of all the elements of K except 0, 1, 2, 4/3, 2/3 if the characteristic of K is not 3. If the characteristic of K is 3, let K' be the set consisting of all the elements of K except 0, 1, 2. Then the cardinality of K' is still $\aleph_0$ (greater than $\aleph_0$). For the given dimension $n=10$, on K' we define a relation $\alpha\sim\beta$ if and only if $L_\alpha\cong L_\beta$ where L_α and L_β are nilpotent Lie algebras of dimension 10 over K as constructed in Lemma 2. Clearly, this relation is an equivalence relation. Hence, K' is partitioned into countably infinitely (uncountably) many pairwise disjoint classes since by Lemma 2 each class consists of only a finite number of elements. The nilpotent Lie algebras corresponding to any two different classes are nonisomorphic. Hence, there are countably infinitely (uncountably) many nonisomorphic nilpotent Lie algebras of class 3 and of dimension 10 over K.

For $n>10$, let $n=10+m$ and construct the direct sum of nilpotent Lie algebras $L'_\alpha=L_\alpha\oplus M$ where M is an abelian Lie algebra of dimension m, and L_α is the nilpotent Lie algebra of dimension 10 as constructed in Lemma 2. Then $L'_\alpha\cong L'_\beta$ if and only if $L_\alpha\cong L_\beta$. Theorem 1 follows.

Now the proof of Theorem 2: Let $L_\alpha=((x_1, \cdots, x_8, y_1, y_2))$ be defined as in Lemma 2, let $N_\alpha=((x_1, \cdots, x_8, y_1, y_2, z_\alpha))$ where the multiplications of x_i's and y_j's are defined the same as in L_α, and $[z_\alpha, x_i]=x_i$, $i=1, 2, \cdots, 8$, and $[z_\alpha, y_j]=2y_j$, $j=1, 2$. Then N_α is a solvable not nilpotent Lie algebra of dimension 11 over K. $N_\alpha\cong N_\beta$ if and only if $L_\alpha\cong L_\beta$. By Theorem 1, there are countably infinitely (uncountably) many nonisomorphic solvable not nilpotent Lie algebras of dimension 11 over K.

For any given dimension $n' > 11$, let $n' = 11 + m$ and construct the direct sum of solvable Lie algebras $N'_\alpha = N_\alpha \oplus M$ where M is an abelian Lie algebra of dimension m. Then $N'_\alpha \cong N'_\beta$ if and only if $N_\alpha \cong N_\beta$ and Theorem 2 follows.

A field R is said to be an S-field (S'-field) if R contains a set S (S') of countably infinitely (uncountably) many elements which are algebraically independent over a subfield Q of R. Clearly, an S'-field is also an S-field. Although Theorem A proved in [1] is over the real number field, it still holds if the real number field is replaced by an S'-field. In addition, we have the following statement: There exist countably infinitely many nonisomorphic nilpotent Lie algebras of class 3 over an S-field for any given dimension $n \geqq 10$. The proof is practically the same as the one in [1]. Certainly, this statement is an immediate consequence of Theorem 1. However, the nilpotent Lie algebras constructed in the proofs are different; in the former case, the center always coincides with the commutator. In the later case, when the dimension of the Lie algebra is odd, the center properly contains the commutator.

3. **Nilpotent associative algebras.** In [2, Theorem 16, p. 72] it is shown that for any integer $n \geqq 6$, there exist infinitely many nonisomorphic commutative nilpotent associative algebras of class 3 and of dimension n over an infinite field F. It seems natural to ask whether there exist infinitely many nonisomorphic nilpotent associative algebras of dimension lower than 6. The method used in [2] does not seem to work when the dimension is lower than 6. Since a nilpotent Lie algebra of class 3 is an associative algebra, we may apply the method used in [1] to prove the following:

THEOREM 3. *Let F be an S-field (S'-field) then there exist countably infinitely (uncountably) many nonisomorphic nilpotent associative algebras of class 3 over F for any given dimension* $\geqq 5$.

Let U be a subfield of a field R. An associative algebra $\mathfrak{A}$ over R is said to be an U-algebra if its structure constants with respect to some basis of $\mathfrak{A}$ lie in U.

Let E be a subfield of F and $\mathfrak{A}$ be an associative algebra of dimension $m+n$ over F defined by a basis $(x_1, \cdots, x_m, y_1, \cdots, y_n)$ with a bilinear multiplication $x_j x_k = \sum_{i=1}^n c_{jk}^i y_i$, $j, k = 1, 2, \cdots, m$, and all other products are zero where $c_{jk}^i \in F$. Then $\mathfrak{A}$ is a nilpotent associative algebra of class 3.

LEMMA 3. *If the numbers c_{jk}^i, $1 \leqq i \leqq n$, $1 \leqq j$, $k \leqq m$ are algebraically independent over E, and if $mn^2 > n^2 + m^2$, then $\mathfrak{A}$ is not an E-algebra.*

We omit the proofs of Lemma 3 and Theorem 3 here since they are practically the same as the proofs in [1]. Although by Theorem 16 in [2] we only need to prove for the case of $\mathfrak{A}$ having dimension 5, (i.e., $m=3$ and $n=2$) our construction shows that the annihilator of $\mathfrak{A}$ coincides with $\mathfrak{A}^2$, and $\mathfrak{A}$ is not commutative.

In particular, we have

COROLLARY. *There exist uncountably many nonisomorphic nilpotent associative algebras of class* 3 *over the real or complex number field for any given dimension* $\geqq 5$.

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