HOMOGENEOUS PRIMARY ABELIAN GROUPS

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A primary abelian group will be said to be homogeneous if it is either divisible or a direct sum of cyclic groups of a fixed order p^n. Kaplansky [**1**, p. 66] states that if G is a homogeneous primary group, then for any two subgroups S and T, there is an automorphism of G carrying S *into* T if and only if S and T are isomorphic, and G/S and G/T are isomorphic. It is the purpose of this paper to show that homogeneous primary groups are characterized by a property which is not far removed from Kaplansky's statement. Finally, a counterexample to Kaplansky's statement will be provided.

Throughout the paper all groups will be p-primary abelian groups. $G[p^n]$ will denote, as usual, the subgroup of elements x of G such that $p^n x = 0$.

THEOREM. *G is a homogeneous p-group if and only if for all subgroups S and T of G the two conditions are equivalent*:

(a) *There is an automorphism of G sending S onto T.*

(b) *S and T are isomorphic and $G[p]/S[p]$ is isomorphic to $G[p]/T[p]$.*

PROOF. Since $G[p]$ is a characteristic subgroup of any primary group G, it is always the case that (a) implies (b). Suppose then that G is homogeneous and S and T are subgroups of G satisfying condition (b). If G is divisible, let E and E' be injective hulls of S and T, respectively, in G. Then $S[p]=E[p]$, $T[p]=E'[p]$, and $G=E\oplus H=E'\oplus K$ for some divisible subgroups H and K of G. Any isomorphism between S and T extends to an isomorphism between E and E'. Furthermore,

$$H[p]\cong G[p]/E[p] = G[p]/S[p]\cong G[p]/T[p]\cong K[p],$$

and since H and K are divisible it follows that H and K are isomorphic. Hence any isomorphism of S into T extends to an automorphism of G.

If, on the other hand, G is a direct sum of cyclic groups of fixed order p^n, then G can be imbedded in its injective hull D so that $G[p]=D[p]$ and $G=D[p^n]$. Now S and T satisfy (b) in D, which is divisible; hence any isomorphism between S and T extends to an automorphism of D. Since $G=D[p^n]$ is a characteristic subgroup of D, this automorphism induces an automorphism of G. Hence if G is homogeneous, conditions (a) and (b) are equivalent.

Received by the editors April 7, 1969.

To prove the converse, suppose G is a primary abelian group with the required property. Let x and y be any two elements in $G[p]$, and let S be the cyclic subgroup generated by x and T that generated by y. Then S and T are isomorphic, as are $G[p]/S[p]$ and $G[p]/T[p]$. Hence there is an automorphism of G sending S onto T. It follows that x and y have the same height in G. Thus all elements of $G[p]$ have the same height in G. If this common height is infinite, G is divisible; otherwise, G is a direct sum of cyclics of the same order p^n. G is therefore seen to be homogeneous and the proof is complete.

REMARK. It is easy to see that the theorem could be restated: A necessary and sufficient condition for a primary group G to be homogeneous is that for each pair of isomorphic subgroups S and T of G such that $G[p]/S[p]$ is isomorphic to $G[p]/T[p]$, each isomorphism of S and T is induced by an automorphism of G.

COROLLARY. *Let G and H be isomorphic homogeneous groups with isomorphic subgroups S and T respectively. Each isomorphism of S onto T extends to an isomorphism of G onto H if and only if $G[p]/S[p]$ is isomorphic to $H[p]/T[p]$.*

The example which was promised earlier is obtained by taking $G=Z(p^\infty)\oplus G'$, where G' is divisible of infinite rank, $S=G[p]$ and $T=G'[p]$. Then S and T are isomorphic, as are G/S and G/T. But no automorphism sends S into T. Conversely, if G and S are as above and $T=G$, then each automorphism of G sends S into T, while G/S and G/T are not isomorphic.

REFERENCE

1. I. Kaplansky, *Infinite abelian groups*, Univ. of Michigan Press, Ann Arbor, Mich., 1954. MR 16, 444.

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