EXTENDING COMMUTING FUNCTIONS

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Let function denote a continuous mapping of a subset of $[0, 1]$ into $[0, 1]$. The proposition in this note offers a method of extending certain commuting functions on closed sets to commuting functions on an interval preserving orbital properties.

DEFINITION. If A is a subset of $[0, 1]$, then u, v will be called *consecutive in A* provided u and v are in A and $A \cap [u, v] = \{u, v\}$; the set of points in $[0, 1]$ which are not in A will be called the *complement of A*. If f is a function defined on A, a closed subset of $[0, 1]$, then: for each positive integer n, let f^n denote the function defined inductively, $f^1 = f$ and $f^{n+1} = f^n f$ (i.e. $f^{n+1}(x) = f^n(f(x))$ for each x in A); let f^* denote the function mapping $[0, 1]$ into itself defined, $f^*|_A = f$ and $f^*|_B$ is linear for every component B of the complement of A; and let *f jump b* provided b is in $f(A)$ and there exist a, a' in A such that b is in $[f(a), f(a')]$ but not in $f([a, a'])$ (i.e. not in $f(A \cap [a, a']))$.

LEMMA. *Let a, b, c, d, u, and v be in $[0, 1]$ such that $a \neq b$, $c \neq d$, and $u \neq v$. Let f be a nowhere constant function mapping $[a, b]$ into $[0, 1]$ and let g be a nowhere constant function mapping $[c, d]$ into $[0, 1]$ such that the four intervals $f([a, b])$, $g([c, d])$, $[f(a), f(b)]$, and $[g(c), g(d)]$ coincide (and $f(a) = g(c)$). Then there exist nowhere constant functions h and j, h mapping $[u, v]$ onto $[c, d]$, j mapping $[u, v]$ onto $[a, b]$ such that: $fj = gh$; $h(u) = c$, $j(u) = a$, $h(v) = d$, and $j(v) = b$.*

PROOF. See [**3**].

PROPOSITION. *Let A be a closed subset of $[0, 1]$, and let f and g be functions without jumps mapping A into A which have coincident ranges containing 0 and 1. For any a, a' consecutive in A assume that $f([g(a), g(a')])$ and $g([f(a), f(a')])$ are subsets of $[fg(a), fg(a')]$, and for any b, b' in A with $b \neq b'$ assume that if $f([b, b'])$ is a singleton then $g^{*-1}([b, b'])$ is a subset of A, and if $g([b, b'])$ is a singleton then $f^{*-1}([b, b'])$ is a subset of A. Then f and g can be extended to functions $\bar{f}$ and $\bar{g}$ mapping $[0, 1]$ onto $[0, 1]$ such that $\bar{f}\bar{g} = \bar{g}\bar{f}$ and for any a, a' consecutive in A, $[f(a), f(a')] = \bar{f}([a, a'])$ and $[g(a), g(a')] = \bar{g}([a, a'])$.*

PROOF. For every natural number n, define the set A_n and the functions f_n and g_n inductively on n. Let $A_0 = f(A)$, $f_0 = f^*$, and $g_0 = g^*$. Let n be a natural number, and if A_n, f_n, and g_n have been defined,

Received by the editors June 17, 1968.

then define $A_{n+1}=f_n^{-1}(A_n)$, and for every u, v consecutive in A_{n+1} let $f_{n+1}|_{[u,v]}$ and $g_{n+1}|_{[u,v]}$ be defined by the above lemma such that $f_n g_{n+1}|_{[u,v]}=g_n f_{n+1}|_{[u,v]}$, $f_{n+1}(u)=f_n(u)$, $f_{n+1}(v)=f_n(v)$, $f_{n+1}([u, v])=f_n([u,v])$, $g_{n+1}(u)=g_n(u)$, $g_{n+1}(v)=g_n(v)$, and $g_{n+1}([u,v])=g_n([u,v])$. It is easy to check inductively that the hypotheses of the lemma are satisfied when the lemma is applied. If $A_{n+1}\cap[f_n(u), f_n(v)]$ is not empty then it can be assumed that $f_{n+1}^{-1}(A_{n+1})\cap[(2u+v)/3, (u+2v)/3]$ is not empty; similarly if $A_{n+1}\cap[g_n(u), g_n(v)]$ is not empty then it can be assumed that $g_{n+1}^{-1}(A_{n+1})\cap[(2u+v)/3, (u+2v)/3]$ is not empty. For each n, let f_n' denote the union of all sets of the form $[u, v]\times f_n([u, v])$ and g_n' denote the union of all sets of the form $[u, v]\times g_n([u, v])$ where u, v are consecutive in A_n. For each n, f_n' and g_n' are closed connected subsets of $[0, 1]\times[0, 1]$; also $f_n'\supseteq f_{n+1}'$ and $g_n'\supseteq g_{n+1}'$. Let $f'=\bigcap_n f_n'$, $g'=\bigcap_n g_n'$, and $A'=\bigcup_n A_n$.

For an a in $[0, 1]$ let $(\{a\}\times[0, 1])\cap f'=\{a\}\times[b, c]$. If $b\neq c$, then $\{a\}$ and $[b, c]$ must be subsets of the closure of the complement of A'. Hence there exists a natural number n such that $[b, c]$ is a subset of the closure of the complement of A_n, and there exists d, e consecutive in A_{n+1} such that $d\neq e$, a is between d and e, and $[d, e]\times[b, c]\subset f'$. Similarly if $\{a\}\times[b, c]\subseteq g'$ with $b\neq c$, then there exist d, e with $d\neq e$, a between d and e, and $[d, e]\times[b, c]\subseteq g'$. Now define functions $\bar{f}$ and $\bar{g}$: for each natural number n, let $\bar{f}|_{A_n}=f_n|_{A_n}$ and let $\bar{g}|_{A_n}=g_n|_{A_n}$; $\bar{f}|_{A'}$ and $\bar{g}|_{A'}$ can be extended to continuous functions (whose graphs are in f' and g' respectively) on $A'-$, the closure of A'; let $\bar{f}=\bar{f}|_{A'-}*$ and $\bar{g}=\bar{g}|_{A'-}*$. Now $\bar{f}$ and $\bar{g}$ satisfy the proposition.

COROLLARY. *There exist commuting functions mapping* $[0, 1]$ *onto itself which do not have a common fixed point.*

PROOF. Let $A=\{n/11 | n=0, \cdots, 11\}$. Let $f(n/11)$ be 0 when $n=0, 8$; be 3/11 when $n=1, 7, 9$; be 8/11 when $n=2, 4, 6, 10$; and be 1 otherwise. Let $g(n/11)$ be 0 when $n=3, 11$; be 3/11 when $n=2, 4, 10$; be 8/11 when $n=1, 5, 7, 9$; and be 1 otherwise. Essentially this same example was constructed in [1] and [2].

COROLLARY. *There exist commuting functions mapping* $[0, 1]$ *to itself which have an open set which is minimally connected and invariant under the functions.*

James T. Joichi suggested the following rewording of the proposition which is sufficient for the two preceding corollaries:

COROLLARY. *Let A be a closed nowhere dense subset of* $[0, 1]$, *and let f and g be functions without jumps mapping A into A which have coinci-*

dent ranges containing 0 *and* 1. *For any* a, a' *consecutive in* A *assume that* $f([g(a), g(a')])$ *and* $g([f(a), f(a')])$ *are subsets of* $[fg(a), fg(a')]$. *Then* f *and* g *can be extended to functions* $\bar{f}$ *and* $\bar{g}$ *mapping* $[0, 1]$ *onto* $[0, 1]$ *such that* $\bar{f}\bar{g}=\bar{g}\bar{f}$ *and for any* a, a' *consecutive in* A, $[f(a), f(a')] = \bar{f}([a, a'])$ *and* $[g(a), g(a')] = \bar{g}([a, a'])$.

References

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