EXTENDING COMMUTING FUNCTIONS

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Let function denote a continuous mapping of a subset of \([0, 1]\) into \([0, 1]\). The proposition in this note offers a method of extending certain commuting functions on closed sets to commuting functions on an interval preserving orbital properties.

DEFINITION. If \(A\) is a subset of \([0, 1]\), then \(u, v\) will be called consecutive in \(A\) provided \(u\) and \(v\) are in \(A\) and \(A \cap [u, v] = \{u, v\}\); the set of points in \([0, 1]\) which are not in \(A\) will be called the complement of \(A\). If \(f\) is a function defined on \(A\), a closed subset of \([0, 1]\), then:

for each positive integer \(n\), let \(f^n\) denote the function defined inductively, \(f^1 = f\) and \(f^{n+1} = f \circ f^n\) (i.e. \(f^{n+1}(x) = f^n(f(x))\) for each \(x\) in \(A\)); let \(f^*\) denote the function mapping \([0, 1]\) into itself defined, \(f^*|_{[0, 1]}\) is linear for every component of the complement of \(A\); and let \(f\) jump \(b\) provided \(b\) is in \(f(A)\) and there exist \(a, a'\) in \(A\) such that \(b\) is in \([f(a), f(a')]\) but not in \((f([a, a']))\) (i.e. not in \((f(A \cap [a, a']))\)).

Lemma. Let \(a, b, c, d, u,\) and \(v\) be in \([0, 1]\) such that \(a \neq b, c \neq d,\) and \(u \neq v\). Let \(f\) be a nowhere constant function mapping \([a, b]\) into \([0, 1]\) and let \(g\) be a nowhere constant function mapping \([c, d]\) into \([0, 1]\) such that the four intervals \(f([a, b]), g([c, d]), [f(a), f(b)],\) and \([g(c), g(d)]\) coincide (and \(f(a) = g(c)\)). Then there exist nowhere constant functions \(h\) and \(j\), \(h\) mapping \([u, v]\) onto \([c, d]\), \(j\) mapping \([u, v]\) onto \([a, b]\) such that: \(fj = gh; h(u) = c, j(u) = a, h(v) = d,\) and \(j(v) = b\).

Proof. See [3].

Proposition. Let \(A\) be a closed subset of \([0, 1]\), and let \(f\) and \(g\) be functions without jumps mapping \(A\) into \(A\) which have coincident ranges containing \(0\) and \(1\). For any \(a, a'\) consecutive in \(A\) assume that \(f([g(a), g(a')])\) and \(g([f(a), f(a')])\) are subsets of \([fg(a), fg(a')],\) and for any \(b, b'\) in \(A\) with \(b \neq b'\) assume that if \(f([b, b'])\) is a singleton then \(g^{*^{-1}}([b, b'])\) is a subset of \(A\), and if \(g([b, b'])\) is a singleton then \(f^{*^{-1}}([b, b'])\) is a subset of \(A\). Then \(f\) and \(g\) can be extended to functions \(\tilde{f}\) and \(\tilde{g}\) mapping \([0, 1]\) onto \([0, 1]\) such that \(\tilde{f} \tilde{g} = \tilde{g} \tilde{f}\) and for any \(a, a'\) consecutive in \(A\), \([f(a), f(a')] = \tilde{f}([a, a'])\) and \([g(a), g(a')] = \tilde{g}([a, a']).\)

Proof. For every natural number \(n\), define the set \(A_n\) and the functions \(f_n\) and \(g_n\) inductively on \(n\). Let \(A_0 = f(A), f_0 = f^*,\) and \(g_0 = g^*.\) Let \(n\) be a natural number, and if \(A_n, f_n,\) and \(g_n\) have been defined,
then define $A_{n+1} = f_n^{-1}(A_n)$, and for every $u, v$ consecutive in $A_{n+1}$ let $f_{n+1} \mid [u, v]$ and $g_{n+1} \mid [u, v]$ be defined by the above lemma such that $f_{n+1}(u) = f_n(u)$, $f_{n+1}(v) = f_n(v)$, $f_{n+1}([u, v]) = f_n([u, v])$, $g_{n+1}(u) = g_n(u)$, $g_{n+1}(v) = g_n(v)$, and $g_{n+1}([u, v]) = g_n([u, v])$. It is easy to check inductively that the hypotheses of the lemma are satisfied when the lemma is applied. If $A_{n+1} \cap [f_n(u), f_n(v)]$ is not empty then it can be assumed that $f_{n+1}^{-1}(A_{n+1}) \cap [(2u+v)/3, (u+2v)/3]$ is not empty; similarly if $A_{n+1} \cap [g_n(u), g_n(v)]$ is not empty then it can be assumed that $g_{n+1}^{-1}(A_{n+1}) \cap [(2u+v)/3, (u+2v)/3]$ is not empty. For each $n$, let $/n$ denote the union of all sets of the form $[u, v] \times f_n([u, v])$ and $g_n'$ denote the union of all sets of the form $[u, v] \times g_n([u, v])$ where $u, v$ are consecutive in $A_n$. For each $n$, $/n$ and $g_n'$ are closed connected subsets of $[0, 1] \times [0, 1]$; also $/n \supseteq /n+1$ and $g_n' \supseteq g_n'+1$. Let $/ = \cap_n /n$, $g' = \cap_n g_n'$, and $A' = \cup_n A_n$.

For an $a$ in $[0, 1]$ let $\{a\} \times [0, 1] \cap / = \{a\} \times [b, c]$. If $b \neq c$, then $\{a\}$ and $[b, c]$ must be subsets of the closure of the complement of $A'$. Hence there exists a natural number $n$ such that $[b, c]$ is a subset of the closure of the complement of $A_n$, and there exists $d, e$ consecutive in $A_{n+1}$ such that $d \neq e$, $a$ is between $d$ and $e$, and $[d, e] \times [b, c] \subseteq /$. Similarly if $\{a\} \times [b, c] \subseteq g'$ with $b \neq c$, then there exist $d, e$ with $d \neq e$, $a$ between $d$ and $e$, and $[d, e] \times [b, c] \subseteq g'$. Now define functions $/f$ and $g$; for each natural number $n$, let $/f \mid A_n = f_n \mid A_n$ and let $g \mid A_n = g_n \mid A_n$; $/f \mid A'$ and $g \mid A'$ can be extended to continuous functions (whose graphs are in $/f'$ and $g'$ respectively) on $A'$, the closure of $A'$; let $/f = /f \mid A'$ and $g = g \mid A'$. Now $/f$ and $g$ satisfy the proposition.

**Corollary.** There exist commuting functions mapping $[0, 1]$ onto itself which do not have a common fixed point.

**Proof.** Let $A = \{n/11 \mid n = 0, \ldots, 11\}$. Let $f(n/11)$ be 0 when $n = 0, 8$; be $3/11$ when $n = 1, 7, 9$; be $8/11$ when $n = 2, 4, 6, 10$; and be 1 otherwise. Let $g(n/11)$ be 0 when $n = 3, 11$; be $3/11$ when $n = 2, 4, 10$; be $8/11$ when $n = 1, 5, 7, 9$; and be 1 otherwise. Essentially this same example was constructed in [1] and [2].

**Corollary.** There exist commuting functions mapping $[0, 1]$ to itself which have an open set which is minimally connected and invariant under the functions.

James T. Joichi suggested the following rewording of the proposition which is sufficient for the two preceding corollaries:

**Corollary.** Let $A$ be a closed nowhere dense subset of $[0, 1]$, and let $f$ and $g$ be functions without jumps mapping $A$ into $A$ which have coinci-
dent ranges containing 0 and 1. For any \( a, a' \) consecutive in \( A \) assume that \( f([g(a), g(a')]) \) and \( g([f(a), f(a')]) \) are subsets of \([fg(a), fg(a')]\). Then \( f \) and \( g \) can be extended to functions \( \tilde{f} \) and \( \tilde{g} \) mapping \([0, 1]\) onto \([0, 1]\) such that \( \tilde{f} \tilde{g} = \tilde{g} \tilde{f} \) and for any \( a, a' \) consecutive in \( A \), \([f(a), f(a')] = \tilde{f}([a, a']) \) and \([g(a), g(a')] = \tilde{g}([a, a']) \).

References


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