For a $W^{*}$ algebra the following two properties are partial forms of commutativity.

**Definition 1.** A $W^{*}$ algebra $\mathcal{A}$ has property $L$ if there are unitary operators $U_{n} \in \mathcal{A}$ such that $U_{n} \to 0$ weakly and such that for every $A \in \mathcal{A}$, $U_{n}A U_{n}^{*} \to A$ strongly.

**Definition 2.** A $W^{*}$ algebra $\mathcal{A}$ has property $A.A.$ if there are *-automorphisms $\phi_{n}$ of $\mathcal{A}$ such that for every pair $S, T \in \mathcal{A}$, $[S, \phi_{n}(T)] \to 0$ strongly, where $[S, T] = ST - TS$.

Definition 1 is due to Pukánszky [3] and Definition 2 is essentially due to Sakai [4].

Let $H$ denote a separable Hilbert space and let $B(H)$ denote the algebra of all bounded operators on $H$. It is our aim to prove that $B(H)$ has neither property $A.A.$ nor property $L$.

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**Lemma 3.** Let $V_{n}$ be a sequence of unitary operators such that $V_{n}^{*}AV_{n} \to A$ strongly for each $A \in B(H)$. Then there is a subsequence (again called $V_{n}$) and a scalar $\lambda$ such that $|\lambda| = 1$ and $V_{n} \to \lambda I$ strongly.

**Proof.** Clearly $A V_{n} - V_{n}A \to 0$ strongly. By the weak compactness of the unit ball of operators we may assume that $V_{n}$ (or a subsequence) converges weakly to some $V$. It follows readily that $AV = VA$ for all $A \in B(H)$ and hence that $V = \lambda I$, $|\lambda| \leq 1$.

Let $\{x_{n}\}$ be an orthonormal basis for $H$ and define an operator $A$ such that $Ax_{1} = x_{1}$, $Ax_{j} = 0$, $j > 1$. A simple calculation shows that

$$((V_{n}^{*}AV_{n} - A)x_{1}, x_{1}) = |(V_{n}x_{1}, x_{1})|^{2} - 1 \to 0.$$  

Since clearly $(V_{n}x_{1}, x_{1}) \to \lambda$, we see that $|\lambda|^{2} = 1$. Thus $|\lambda| = 1$. Since the $V_{n}$ are unitary and converge weakly to the unitary $\lambda I$, it follows that $V_{n} \to \lambda I$ strongly. Q.E.D.

**Corollary 4.** $B(H)$ does not have property $L$.

**Proof.** Use Lemma 3 putting $V_{n} = U_{n}^{*}$. Q.E.D.

**Theorem 5.** $B(H)$ does not have property $A.A.$

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Proof. Since it is well known that every \(*\)-automorphism of \(B(H)\) is inner (see [1]) we may assume that \(\phi_n(A) = U_n A U_n^*\) for some unitary \(U_n\). Suppose therefore that \(B(H)\) has property \(A.A\) and that \(\{U_n\}\) is the corresponding sequence of unitaries demonstrating the property. It follows that for each pair \(A, B \in B(H)\) we have \(U_n A U_n^* B - B U_n A U_n^* \to 0\) strongly.

Let \(E\) denote the orthogonal projection on the span of \(\{x_1, x_2, \cdots\}\), and let \(F = I - E\). Then there is a unitary \(V\) such that \(E = VFV^*\), and there are operators \(P, Q, R, S\) such that \(I = SER = PFQ\). Let \(\{V_m\}\) be a countable set of unitaries such that \(E, F, P, Q, R, S, V\) are linear combinations of the \(V_m\) and such that \(\{V_m\}\) is closed under multiplication and taking adjoints. Let \(\mathcal{A}\) be the involutive algebra generated by the \(\{V_m\}\). We may assume that the \(U_n\) are such that the strong limit of the sequence \(U_n V_m U_n^* = \lambda_m I\) exists for each \(V_m\), by Lemma 3 and the Cantor diagonal process. Then it is readily verified that if \(A \in \mathcal{A}\), \(A = \sum_{m=1}^{N} a_m V_m\), we may define \(\phi: \mathcal{A} \to \mathcal{C}\) by \(\phi(A) = \lambda\) where strong limit \(U_n A U_n^* = \lambda I\). It is easy to check that \(\phi\) is linear, multiplicative, and \(\phi(I) = 1\). Hence \(\phi(E) = 0\) or \(\phi(E) = 1\). But \(\phi(F) = \phi(V) \phi(E) \phi(V^*)\), and therefore \(\phi(F) = \phi(E)\). Therefore \(\phi(I) = \phi(E) + \phi(F)\), so that \(\phi(I) = 0\) or \(\phi(I) = 2\), both contradictions. It follows that \(B(H)\) does not have property \(A.A\). Q.E.D.

This proof is essentially the proof of Theorem 4.1 in [2].

References

4. S. Sakai, Asymptotically Abelian \(II_1\)-factors (to appear).

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