ON DECOMPOSABLE OPERATORS

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Let E be a Banach space, and let $\mathcal{L}(E)$ be the set of continuous linear mappings of E into E. $T\in\mathcal{L}(E)$ has the *single-valued extension property* if, for any open subset Q of $\boldsymbol{C}$ and any analytic function f with domain Q and values in E such that $(\lambda-T)f(\lambda)=0$ for $\lambda\in Q$, $f=0$. In this case, for every $x\in E$, there exists a maximally defined analytic function φ_x with values in E such that $(\lambda-T)\varphi_x(\lambda)=x$ for all λ in the domain of φ_x. The complement of the domain of φ_x in $\boldsymbol{C}$ will be denoted by $\mathrm{sp}_T(x)$. For any $F\subset\boldsymbol{C}$ let

$$\mathfrak{M}_T(F) = \{x \in E : \mathrm{sp}_T(x) \subset F\}.$$

From here on E will be a fixed Banach space, and T will be a fixed element of $\mathcal{L}(E)$. The symbol $\boldsymbol{N}$ will denote the set $\{2, 3, \cdots\}$.

A closed subspace D of E, invariant under T, is called *spectral maximal* if $Z\subset D$ for any other closed subspace Z of E invariant under T such that $\mathrm{sp}(T|Z)\subset\mathrm{sp}(T|D)$.

DEFINITION. Let $n\in\boldsymbol{N}$. T is *n-decomposable* [resp. *strongly n-decomposable*] if, for any open cover $(G_i)_{1\leq i\leq n}$ of $\boldsymbol{C}$ consisting of n (not necessarily distinct) sets, there exists a corresponding family $(D_i)_{1\leq i\leq n}$ of spectral maximal spaces such that $\mathrm{sp}(T|D_i)\subset G_i(1\leq i\leq n)$ and $E=D_1+\cdots+D_n$ [resp. $D=D\cap D_1+\cdots+D\cap D_n$ for every spectral maximal space D]. T is [*strongly*] *decomposable* if T is [strongly] n-decomposable for every $n\in\boldsymbol{N}$.

We will need the following facts. References [1] and [2] can be consulted for proofs.

(1) If T is n-decomposable, then T has the single-valued extension property and $\mathfrak{M}_T(F)$ is a spectral maximal space with $\mathrm{sp}(T|\mathfrak{M}_T(F))\subset F\cap\mathrm{sp}(T)$ for every closed subset F of $\boldsymbol{C}$. If D is a spectral maximal space, $D=\mathfrak{M}_T(\mathrm{sp}(T|D))$.

(2) T is strongly n-decomposable if and only if $T|D$ is n-decomposable for every spectral maximal space D.

THEOREM. *If $T\in\mathcal{L}(E)$ is strongly 2-decomposable, then $\mathfrak{M}_T(G_1\cup G_2)=\mathfrak{M}_T(G_1)+\mathfrak{M}_T(G_2)$ for any open subsets G_1 and G_2 of $\boldsymbol{C}$.*

PROOF. It is enough to show that $\mathfrak{M}_T(G_1\cup G_2)\subset\mathfrak{M}_T(G_1)+\mathfrak{M}_T(G_2)$. To this end let $x\in\mathfrak{M}_T(G_1\cup G_2)$. Then $D=\mathfrak{M}_T(\mathrm{sp}_T(x))$ is spectral maximal, $T|D$ is 2-decomposable and, since $\mathrm{sp}(T|D)\subset\mathrm{sp}_T(x)$,

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$\{G_1, G_2\}$ is an open cover of $\mathrm{sp}(T|D)$. Therefore there exist spectral maximal spaces D_1 and D_2 of $T|D$ such that $\mathrm{sp}(T|D_i)\subset G_i$ $(i=1, 2)$ and $D=D_1+D_2\subset\mathfrak{M}_T(G_1)+\mathfrak{M}_T(G_2)$. Since $x\in D$, the result follows.

COROLLARY. *Every strongly 2-decomposable operator T is strongly decomposable.*

PROOF. By (2) above it is enough to show that T is decomposable. Let $(G_i)_{1\leqq i\leqq n}$ be an open cover of $\mathrm{sp}(T)$, and let $(H_i)_{1\leqq i\leqq n}$ be an open cover of $\mathrm{sp}(T)$ such that $\overline{H}_i\subset G_i$ for every $1\leqq i\leqq n$. For each $1\leqq i\leqq n$, $D_i=\mathfrak{M}_T(\overline{H}_i)$ is spectral maximal, $\mathrm{sp}(T|D_i)\subset\overline{H}_i\subset G_i$, and

$$\begin{aligned} E &= \mathfrak{M}_T(H_1\cup\cdots\cup H_n) \\ &= \mathfrak{M}_T(H_1)+\cdots+\mathfrak{M}_T(H_n)\subset D_1+\cdots+D_n\subset E. \end{aligned}$$

(This last step is justified by the above theorem.)

OPEN QUESTIONS. 1. Is every 2-decomposable operator strongly 2-decomposable?

2. Is every 2-decomposable operator decomposable?

An affirmative answer to Question 1 implies affirmative answers to Question 2 and Open Question (b) of [2, p. 217], which in turn provides an interesting characterization of decomposable operators (by the above theorem). The operators introduced by the author [3], which include operators studied by Colojoara, Dunford, Foias, Kantorovitz, and Maeda, are strongly decomposable.

REFERENCES

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