ON DECOMPOSABLE OPERATORS

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Let $E$ be a Banach space, and let $\mathcal{L}(E)$ be the set of continuous linear mappings of $E$ into $E$. $T \in \mathcal{L}(E)$ has the single-valued extension property if, for any open subset $Q$ of $\mathbb{C}$ and any analytic function $f$ with domain $Q$ and values in $E$ such that $(\lambda - T)f(\lambda) = 0$ for $\lambda \in Q$, $f = 0$. In this case, for every $x \in E$, there exists a maximally defined analytic function $\varphi_x$ with values in $E$ such that $(\lambda - T)\varphi_x(\lambda) = x$ for all $\lambda$ in the domain of $\varphi_x$. The complement of the domain of $\varphi_x$ in $\mathbb{C}$ will be denoted by $\text{sp}_T(x)$. For any $F \subseteq \mathbb{C}$ let

$$\mathfrak{M}_T(F) = \{x \in E : \text{sp}_T(x) \subseteq F\}.$$ 

From here on $P$ will be a fixed Banach space, and $T$ will be a fixed element of $\mathcal{L}(E)$. The symbol $\mathbb{N}$ will denote the set $\{2, 3, \cdots \}$.

A closed subspace $D$ of $E$, invariant under $T$, is called spectral maximal if $Z \subseteq D$ for any other closed subspace $Z$ of $E$ invariant under $T$ such that $\text{sp}(T|Z) \subseteq \text{sp}(T|D)$.

Definition. Let $n \in \mathbb{N}$. $T$ is $n$-decomposable [resp. strongly $n$-decomposable] if, for any open cover $(G_i)_{1 \leq i \leq n}$ of $\mathbb{C}$ consisting of $n$ (not necessarily distinct) sets, there exists a corresponding family $(D_i)_{1 \leq i \leq n}$ of spectral maximal spaces such that $\text{sp}(T|D_i) \subseteq G_i (1 \leq i \leq n)$ and $E = D_1 + \cdots + D_n$ [resp. $D = D \cap D_1 + \cdots + D \cap D_n$ for every spectral maximal space $D$]. $T$ is [strongly] decomposable if $T$ is [strongly] $n$-decomposable for every $n \in \mathbb{N}$.

We will need the following facts. References [1] and [2] can be consulted for proofs.

(1) If $P$ is re-decomposable, then $P$ has the single-valued extension property and $\mathfrak{M}_T(F)$ is a spectral maximal space with $\text{sp}(T|\mathfrak{M}_T(F)) \subseteq F \cap \text{sp}(T)$ for every closed subset $F$ of $\mathbb{C}$. If $D$ is a spectral maximal space, $D = \mathfrak{M}_T(\text{sp}(T|D))$.

(2) $T$ is strongly $n$-decomposable if and only if $T|D$ is $n$-decomposable for every spectral maximal space $D$.

Theorem. If $T \in \mathcal{L}(E)$ is strongly 2-decomposable, then $\mathfrak{M}_T(G_1 \cup G_2) = \mathfrak{M}_T(G_1) + \mathfrak{M}_T(G_2)$ for any open subsets $G_1$ and $G_2$ of $\mathbb{C}$.

Proof. It is enough to show that $\mathfrak{M}_T(G_1 \cup G_2) \subseteq \mathfrak{M}_T(G_1) + \mathfrak{M}_T(G_2)$. To this end let $x \in \mathfrak{M}_T(G_1 \cup G_2)$. Then $D = \mathfrak{M}_T(\text{sp}_T(x))$ is spectral maximal, $T|D$ is 2-decomposable and, since $\text{sp}(T|D) \subseteq \text{sp}(T), received by the editors June 4, 1969.
\{G_1, G_2\} is an open cover of sp(T|D). Therefore there exist spectral maximal spaces $D_1$ and $D_2$ of $T|D$ such that $\text{sp}(T|D_i) \subset G_i$ $(i=1, 2)$ and $D = D_1 + D_2 \subset \mathfrak{M}_T(G_1) + \mathfrak{M}_T(G_2)$. Since $x \in D$, the result follows.

**Corollary.** Every strongly 2-decomposable operator $T$ is strongly decomposable.

**Proof.** By (2) above it is enough to show that $T$ is decomposable. Let $(G_i)_{1 \leq i \leq n}$ be an open cover of $\text{sp}(T)$, and let $(H_i)_{1 \leq i \leq n}$ be an open cover of $\text{sp}(T)$ such that $\overline{H}_i \subset G_i$ for every $1 \leq i \leq n$. For each $1 \leq i \leq n$, $D_i = \mathfrak{M}_T(\overline{H}_i)$ is spectral maximal, $\text{sp}(T|D_i) \subset \overline{H}_i \subset G_i$, and

$$E = \mathfrak{M}_T(H_1 \cup \cdots \cup H_n) = \mathfrak{M}_T(H_1) + \cdots + \mathfrak{M}_T(H_n) \subset D_1 + \cdots + D_n \subset E.$$  

(This last step is justified by the above theorem.)

**Open Questions.** 1. Is every 2-decomposable operator strongly 2-decomposable?

2. Is every 2-decomposable operator decomposable?

An affirmative answer to Question 1 implies affirmative answers to Question 2 and Open Question (b) of [2, p. 217], which in turn provides an interesting characterization of decomposable operators (by the above theorem). The operators introduced by the author [3], which include operators studied by Colojoara, Dunford, Foias, Kantorovitz, and Maeda, are strongly decomposable.

**References**


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