Kirby and Siebenmann have recently proved that every boundary-less topological \( n \)-manifold \( (n \geq 5) \) having trivial 4-dimensional \( \mathbb{Z}_2 \)-cohomology can be given a combinatorial triangulation; they have also given an example of a topological manifold which supports no combinatorial triangulation. It is quite possible, however, that the manifold constructed might be triangulable by a complex which is not a combinatorial manifold. The main purpose of this note is to show that if this could be done in such a way that the open simplexes are locally flat, then either the 3- or the 4-dimensional Poincaré conjecture would be false. Let us call such a triangulation \textit{locally flat}; note that we do not require that the closed simplexes be locally flat. Then our main result is:

**Theorem 3.** Suppose \( n \geq 5 \). Then the 3- and 4-dimensional Poincaré conjectures together are equivalent to the conjecture that every locally flat triangulation of a topological \( n \)-manifold is combinatorial.

Let \( X \subset Y \) be metric spaces and suppose \( z \in X \). A monotone sequence \( V_1 \supset V_2 \supset \cdots \) of compact neighborhoods of \( z \) in \( Y \) is said to be a \textit{local homotopy} sequence for \( X \) at \( z \) in \( Y \) if \( \bigcap V_i = \{z\} \) and each inclusion \( V_i \setminus X \) into \( V_k \setminus X \) \((k < j)\) is a homotopy equivalence. If such a neighborhood sequence exists, then the homotopy type of \( V_1 \setminus X \) is called the \textit{local homotopy type} of \( X \) at \( z \) in \( Y \); that the local homotopy type is independent of the choice of defining sequence may be seen as follows: if \( U_i \) is another such sequence, we may find integers \( p, q, r \) such that \( U_r \subset V_q \subset U_p \subset V_1 \). The inclusions form a commutative diagram as follows:

\[
\begin{array}{ccc}
U_p \setminus X & \xrightarrow{\alpha'} & U_p \setminus X \\
\beta' \downarrow & & \downarrow \beta \\
V_q \setminus X & \xrightarrow{\gamma} & V_1 \setminus X \\
\end{array}
\]

Since \( \alpha' \) and \( \alpha \) are homotopy equivalences, with homotopy inverses

Received by the editors April 14, 1969.

1 The author was partially supported by the National Science Foundation, Grant 8615.
\(a'\) and \(\alpha, \beta'\alpha'\) is a right homotopy inverse for \(\gamma\) and \(\alpha\beta\) is a left homotopy inverse for \(\gamma\). Hence \(\gamma\) is a homotopy equivalence.

There are two instances of importance to us where the local homotopy type is defined:

**Proposition 1.** The local homotopy type of \(X\) at \(z\) in \(Y\) is well defined if

(i) \(X\) is a \(k\)-manifold in the \(n\)-manifold \(Y\), and \(X\) is locally flat at \(z\); the local homotopy type is that of the \((n-k-1)\)-sphere;

(ii) \(z\) is an interior point of a simplex \(X\) in a complex \(Y\); the local homotopy type is the homotopy type of the link of \(X\) in \(Y\), \(\text{lk}(X, Y)\).

**Corollary 1.** If \(\sigma^k\) is a simplex of a locally flat triangulation of a topological \(n\)-manifold, then \(\text{lk}(\sigma, K)\) has the homotopy type of \(S^{n-k-1}\).

Part (i) of the proposition reduces to the case \(X = \mathbb{R}^k\), \(Y = \mathbb{R}^n\), \(z = \text{origin}\), and we then define \(V_i\) to be the ball of radius \(1/i\) about \(Z\); \(V_i \setminus \mathbb{R}^k\) then deforms to \(S^{n-1} \setminus S^{k-1}\), which has the homotopy type of \(S^{n-k-1}\).

For part (ii), \(V_1 = \text{st}(X, Y)\) and \(V_i = \text{st}(\sigma, Y^{(i)})\), where \(Y^{(i)}\) is an \(i\)th derived subdivision of \(Y\), with \(X\) originally starred at \(z\). In this case, \(V_1 \setminus X = \text{st}(X, Y) \setminus X\) deformation retracts to \(\text{lk}(X, Y)\).

The corollary then follows from (i) and (ii) by considering the local homotopy type of \(\sigma\) at its barycenter.

**Theorem 1.** The Poincaré conjectures imply that every locally flat triangulation of a topological manifold is combinatorial.

**Proof.** Assume the Poincaré conjectures, and let \(K\) be a triangulation of a topological \(n\)-manifold. Let \(\sigma^k\) be a simplex of maximal dimension such that \(\text{lk}(\sigma, K)\) is not PL homeomorphic to \(S^{n-k-1}\). If such a simplex exists, then \(\text{lk}(\sigma, K)\) must be a combinatorial manifold: if \(\text{lk}(\sigma, K)\) fails to be combinatorial at a vertex \(v\), then \(\text{lk}(v, K) = \text{lk}(v, \text{lk}(\sigma, K))\) is not a PL-sphere, violating the maximality of \(\sigma\). Thus, by Corollary 1, \(\text{lk}(\sigma, K)\) is a homotopy sphere, and hence, by the Poincaré conjecture, a sphere. In other words, there was no such simplex \(\sigma\), so that \(K\) is combinatorial.

**Lemma 1.** If \(K\) is combinatorial at \(\sigma\), then \(\text{int} \sigma\) is locally flat.

**Proof.** The pair \((\text{st}(\sigma, K), \sigma)\) is an unknotted ball pair.

**Lemma 2.** If \(K\) triangulates a topological \(n\)-manifold, then \(\sigma\) is combinatorial at simplexes of dimension \(k \geq n-3\).

**Proof.** For each simplex \(\sigma\) of \(K\), \(\text{lk}(\sigma, K)\) must be a pseudomani-
fold with the homology of a sphere; if \( \dim \sigma \geq n - 3 \), then \( \dim(\text{lk}(\sigma, K)) \leq 2 \), and it is well known that \( \text{lk}(\sigma, K) \) must then be a sphere.

**Theorem 2.** If either the 3-dimensional, or the 4-dimensional Poincaré conjecture is false, then there are locally flat noncombinatorial triangulation of \( S^n \) for all \( n \geq 5 \), and hence for all combinatorial \( n \)-manifolds, \( n \geq 5 \).

**Proof.** We need only show that there would be a locally flat noncombinatorial triangulation of \( S^6 \), since suspension of a locally flat triangulation gives a locally flat triangulation (recall that “locally flat” means locally flat on open simplexes.) If the 4-dimensional Poincaré conjecture fails, the suspension of a fake 4-sphere is a topological 5-sphere (see, for example, [2]), yielding a noncombinatorial triangulation of \( S^6 \). Since vertices are automatically locally flat, and since such a triangulation would be combinatorial at all other simplexes, it would be a locally flat triangulation. If the 3-dimensional Poincaré conjecture were false, there would be a fake 3-sphere \( M^3 \) whose double suspension \( S^1 \ast M^3 \) is a topological 5-sphere [2]. If \( K \) triangulates \( M^3 \), and \( L \) triangulates \( S^1 \), then the join \( L \ast K \) would be a noncombinatorial triangulation of \( S^6 \). \( L \ast K \) would be combinatorial at all simplexes except those belonging to \( L \). Let \( \sigma^1 \) be an arbitrary simplex of \( L \), and let \( x \) be an interior point of \( \sigma^1 \). Then the local homotopy type of \( S^1 \) at \( x \) in \( S^6 = S^1 \ast M^3 \) is just the homotopy type of \( M^3 \), which is simply connected. As a consequence, the circle \( S^1 \) is \( 1 - LC \) at \( x \); the local homotopy type of \( S^1 \) at a vertex \( v \) is the homotopy type of \( S^0 \ast M^3 \), which is also simply connected. Thus \( S^1 \) is a locally nicely imbedded 1-sphere in \( S^6 \), and hence is flat, [1, Theorem 4.2]. Thus \( L \ast K \) is a locally flat, noncombinatorial triangulation of \( S^6 \).

**References**


**University of Georgia**