NONCOMBINATORIAL TRIANGULATIONS AND THE POINCARÉ CONJECTURE

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Kirby and Siebenmann have recently proved that every boundaryless topological n-manifold ($n \geqq 5$) having trivial 4-dimensional Z_2-cohomology can be given a combinatorial triangulation; they have also given an example of a topological manifold which supports no combinatorial triangulation. It is quite possible, however, that the manifold constructed might be triangulable by a complex which is not a combinatorial manifold. The main purpose of this note is to show that if this could be done in such a way that the open simplexes are locally flat, then either the 3- or the 4-dimensional Poincaré conjecture would be false. Let us call such a triangulation *locally flat*; note that we do ***not*** require that the closed simplexes be locally flat. Then our main result is:

THEOREM 3. *Suppose* $n \geqq 5$. *Then the* 3- *and* 4-*dimensional Poincaré conjectures together are equivalent to the conjecture that every locally flat triangulation of a topological n-manifold is combinatorial.*

Let $X \subset Y$ be metric spaces and suppose $z \in X$. A monotone sequence $V_1 \supset V_2 \supset \cdots$ of compact neighborhoods of z in Y is said to be a *local homotopy* sequence for X at z in Y if $\bigcap V_i = \{z\}$ and each inclusion $V_j \setminus X$ into $V_k \setminus X$ ($k < j$) is a homotopy equivalence. If such a neighborhood sequence exists, then the homotopy type of $V_1 \setminus X$ is called the *local homotopy type* of X at z in Y; that the local homotopy type is independent of the choice of defining sequence may be seen as follows: if U_i is another such sequence, we may find integers p, q, r such that $U_r \subset V_q \subset U_p \subset V_1$. The inclusions form a commutative diagram as follows:

$$\begin{tikzcd} U_r \setminus X \arrow[r, "\alpha'"] \arrow[d, "\beta'"'] & U_p \setminus X \arrow[d, "\beta"] \\ V_q \setminus X \arrow[ur, "\gamma"] \arrow[r, "\alpha"'] & V_1 \setminus X \end{tikzcd}$$

Since α' and α are homotopy equivalences, with homotopy inverses

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$\bar{\alpha}'$ and $\bar{\alpha}$, $\beta'\bar{\alpha}'$ is a right homotopy inverse for γ and $\bar{\alpha}\beta$ is a left homotopy inverse for γ. Hence γ is a homotopy equivalence.

There are two instances of importance to us where the local homotopy type is defined:

PROPOSITION 1. *The local homotopy type of X at z in Y is well defined if*

(i) *X is a k-manifold in the n-manifold Y, and X is locally flat at z; the local homotopy type is that of the $(n-k-1)$-sphere;*

(ii) *z is an interior point of a simplex X in a complex Y; the local homotopy type is the homotopy type of the link of X in Y, $\operatorname{lk}(X, Y)$.*

COROLLARY 1. *If σ^k is a simplex of a locally flat triangulation of a topological n-manifold, then $\operatorname{lk}(\sigma, K)$ has the homotopy type of S^{n-k-1}.*

Part (i) of the proposition reduces to the case $X=R^k$, $Y=R^n$, $z=\text{origin}$, and we then define V_i to be the ball of radius $1/i$ about Z; $V_1\backslash R^k$ then deforms to $S^{n-1}\backslash S^{k-1}$, which has the homotopy type of S^{n-k-1}.

For part (ii), $V_1=\operatorname{st}(X, Y)$ and $V_i=\operatorname{st}(z, Y^{(i)})$, where $Y^{(i)}$ is an rth derived subdivision of Y, with X originally starred at z. In this case, $V_1\backslash X=\operatorname{st}(X, Y)\backslash X$ deformation retracts to $\operatorname{lk}(X, Y)$.

The corollary then follows from (i) and (ii) by considering the local homotopy type of σ at its barycenter.

THEOREM 1. *The Poincaré conjectures imply that every locally flat triangulation of a topological manifold is combinatorial.*

PROOF. Assume the Poincaré conjectures, and let K be a triangulation of a topological n-manifold. Let σ^k be a simplex of maximal dimension such that $\operatorname{lk}(\sigma, K)$ is not PL homeomorphic to S^{n-k-1}. If such a simplex exists, then $\operatorname{lk}(\sigma, K)$ must be a combinatorial manifold: if $\operatorname{lk}(\sigma, K)$ fails to be combinatorial at a vertex v, then $\operatorname{lk}(v_\sigma, K)=\operatorname{lk}(v, \operatorname{lk}(\sigma, K))$ is not a PL-sphere, violating the maximality of σ. Thus, by Corollary 1, $\operatorname{lk}(\sigma, K)$ is a homotopy sphere, and hence, by the Poincaré conjecture, a sphere. In other words, there was no such simplex σ, so that K is combinatorial.

LEMMA 1. *If K is combinatorial at σ, then* int σ *is locally flat.*

PROOF. The pair $(\operatorname{st}(\sigma, K), \sigma)$ is an unknotted ball pair.

LEMMA 2. *If K triangulates a topological n-manifold, then σ is combinatorial at simplexes of dimension $k \geqq n-3$.*

PROOF. For each simplex σ of K, $\operatorname{lk}(\sigma, K)$ must be a pseudomanifold with the homology of a sphere; if $\dim \sigma \geqq n-3$, then $\dim(\mathrm{lk}(\sigma, K)) \leqq 2$, and it is well known that $\mathrm{lk}(\sigma, K)$ must then be a sphere.

THEOREM 2. *If either the 3-dimensional, or the 4-dimensional Poincaré conjecture is false, then there are locally flat noncombinatorial triangulation of S^n for all $n \geqq 5$, and hence for all combinatorial n-manifolds, $n \geqq 5$.*

PROOF. We need only show that there would be a locally flat noncombinatorial triangulation of S^5, since suspension of a locally flat triangulation gives a locally flat triangulation (recall that "locally flat" means locally flat on *open* simplexes.) If the 4-dimensional Poincaré conjecture fails, the suspension of a fake 4-sphere is a topological 5-sphere (see, for example, [2]), yielding a noncombinatorial triangulation of S^5. Since vertices are automatically locally flat, and since such a triangulation would be combinatorial at all other simplexes, it would be a locally flat triangulation. If the 3-dimensional Poincaré conjecture were false, there would be a fake 3-sphere M^3 whose double suspension $S^1 * M^3$ is a topological 5-sphere [2]. If K triangulates M^3, and L triangulates S^1, then the join $L * K$ would be a noncombinatorial triangulation of S^5. $L * K$ would be combinatorial at all simplexes except those belonging to L. Let σ^1 be an arbitrary simplex of L, and let x be an interior point of σ^1. Then the local homotopy type of S^1 at x in $S^5 = S^1 * M^3$ is just the homotopy type of M^3, which is simply connected. As a consequence, the circle S^1 is $1-LC$ at x; the local homotopy type of S^1 at a vertex v is the homotopy type of $S^0 * M^3$, which is also simply connected. Thus S^1 is a locally nicely imbedded 1-sphere in S^5, and hence is flat, [1, Theorem 4.2]. Thus $L * K$ is a locally flat, noncombinatorial triangulation of S^5.

REFERENCES

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