ON GROUPS OF ORDER $2^\alpha 3^\beta p^\gamma$ WITH A CYCLIC SYLOW 3-SUBGROUP

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The fundamental classification of N-groups by Thompson in [7][1] yields, among others, the following result: if the order of a nonsolvable finite group is divisible by three primes only, then the primes are: 2, 3 and one from the set $\{5, 7, 13, 17\}$. Thus the problem of classification of all simple groups of order $r^\alpha q^\beta p^\gamma$, r, q and p primes, reduces to the classification of simple groups of order mentioned in the title.

In [1] Brauer solved this problem for the case $p^\gamma = 5$. The author showed in [5] that if one of the Sylow groups of G is cyclic and if it is not the Sylow subgroup of G of least order, then G is isomorphic to one of the groups: $PSL(2, 5)$, $PSL(2, 7)$, $PSL(2, 8)$ and $PSL(2, 17)$. The purpose of this paper is to classify all simple groups of order $2^\alpha 3^\beta p^\gamma$ with a cyclic Sylow 3-subgroup.

THEOREM 1. *Let G be a simple group of order $2^\alpha 3^\beta p^\gamma$, where p is a prime, and suppose that a Sylow 3-subgroup Q of G is cyclic. Then G is isomorphic to one of the groups*: $PSL(2, 5)$, $PSL(2, 7)$, $PSL(2, 8)$ *and* $PSL(2, 17)$.

As a matter of fact, we will prove the following more general result:

THEOREM 2. *Let G be a simple group of order $2^\alpha 3^\beta p^\gamma$, where p is a prime. Suppose that a Sylow subgroup R of G is cyclic and $[N_G(R): C_G(R)] = 2$. Then G is isomorphic to one of the groups*: $PSL(2, 5)$, $PSL(2, 7)$, $PSL(2, 8)$, $PSL(2, 9)$ *and* $PSL(2, 17)$.

Theorem 1 follows easily from Theorem 2. Indeed, it is well known that $[N_G(Q): C_G(Q)] = 1$ or 2. It follows from the simplicity of G and the theorem of Burnside that $[N_G(Q): C_G(Q)] = 2$. Thus the assumptions of Theorem 2 are satisfied and it is well known that the groups mentioned in Theorem 2, PSL(2, 9) excluded, satisfy the assumptions of Theorem 1.

PROOF OF THEOREM 2. Let R be a Sylow r-subgroup of G of order r^δ. Since G is simple $r \neq 2$ and $2^\alpha \geqq 4$. It follows from Proposition 2.1 and Corollary 2.1 in [4] that the principal r-block B_1 of G contains $(r^\delta - 1)/2$ exceptional characters of order $x_0 = a_0 r^\delta - 2\epsilon_0$ and two non-exceptional (ordinary) characters: the principal character 1_G and

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another character X_2 of order $x_2=a_2r^{\delta}+\epsilon_2$, where a_0, a_2 are nonnegative integers and ϵ_0, $\epsilon_2=\pm 1$. Formula (6) in [4] then yields:

$$0 = 1 + \epsilon_0 x_0 + \epsilon_2 x_2 = 1 + \epsilon_0 a_0 r^{\delta} - 2 + \epsilon_2 a_2 r^{\delta} + 1.$$

Therefore $a_0=a_2$, $\epsilon_2=-\epsilon_0$ and consequently

$$x_0 = a_0 r^{\delta} - 2\epsilon_0, \qquad x_2 = a_0 r^{\delta} - \epsilon_0.$$

Thus x_0, r and x_2 are prime to each other, and it follows from our assumptions that one of the following cases holds:

Case A: $x_0=2^{\nu}$, $x_2=u^{\eta}$,

Case B: $x_0=u^{\eta}$, $x_2=2^{\nu}$,

where $\{u, r\}=\{3, p\}$ and ν, η are positive integers. In both cases it follows from the formulas for x_0 and x_2 that

$$u^{\eta} - 2^{\nu} = \epsilon, \qquad \epsilon = \pm 1.$$

If $\eta=1$ then it is well known [3, Theorem 18.4] that u is the order of a Sylow u-subgroup of G. Consequently, all the Sylow subgroups of odd order of G are cyclic and by Theorem 1 of [4] G is isomorphic to one of the groups: PSL(2, 5), PSL(2, 7), PSL(2, 8) and PSL(2, 17). Since these groups satisfy the assumptions of Theorem 2, we are done in the case $\eta=1$.

Now assume that $\eta>1$. Then by [6, Theorem 2, p. 335 and Exercise 1, p. 346] $u=3$, $\eta=2$, $\nu=3$ and $\epsilon=1$. We will deal now separately with Cases A and B.

Case A. It follows from the formulas for x_0 and x_2 that:

$$8 = x_0 = a_0 r^{\delta} - 2\epsilon_0, \qquad 9 = x_2 = x_0 + \epsilon_0$$

hence: $r^{\delta}=5$. It follows then from [1] that G is isomorphic either to PSL(2, 5) or to PSL(2, 9). Since the order of PSL(2, 5) is not divisible by 9, only PSL(2, 9) satisfies the assumptions of Case A.

Case B. The formulas for x_0 and x_2 now yield:

$$9 = x_0 = a_0 r^{\delta} - 2\epsilon_0, \qquad 8 = x_2 = x_0 + \epsilon_0$$

hence $r^{\delta}=7$. Thus B_1 contains the irreducible character X_2 of degree $8<2r=14$ and by Lemma 1 of [2] $C_G(\rho)=R$ for every nonidentity element ρ of R. Consequently, every r-singular element of G is of order r. Lemma 3 of [2] then yields that if B_2 is the 2-block to which X_2 belongs then:

$$\sum X(1)X(\rho) \equiv 0 \pmod{2^{\alpha}} \qquad X \text{ in } B_1 \cap B_2$$

where ρ is any r-singular element of G. Since by Lemma 2 of [2] B_2

is a block of defect $\alpha - 1$ at most, it contains characters of even orders only and therefore $B_1 \cap B_2 = \{X_2\}$. As ρ is conjugate to an element of $R^\#$, it follows from Proposition 2.1 of [4] that

$$X_2(\rho) = \epsilon_2 = -\epsilon_0 = 1.$$

The above summation formula then reads:

$$8 = x_2 \cdot 1 \equiv 0 \pmod{2^\alpha}.$$

Since x_2 divides $o(G)$, the order of G, it follows that $2^\alpha = 8$ and

$$o(G) = 7.8.3^\beta, \qquad \beta \geqq 2.$$

Let T be a Sylow 2-subgroup of G. It is well known that T cannot be the quaternion group. If T is dihedral or Abelian, then by Theorem 2 in [5] G has to be isomorphic to one of the groups PSL(2, 5), PXL(2, 7), PSL(2, 8), PSL(2, 9) and PSL(2, 17). It is easy to check that only PSL(2, 8) has order of the form $7.8.3.^\beta$, $\beta \geqq 2$, and consequently only PSL(2, 8) satisfies the assumptions of Case B. The proof of Theorem 2 is complete.

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