ON GROUPS OF ORDER $2^a 3^b p^c$ WITH A CYCLIC SYLOW 3-SUBGROUP

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The fundamental classification of $N$-groups by Thompson in [7]¹ yields, among others, the following result: if the order of a nonsolvable finite group is divisible by three primes only, then the primes are: 2, 3 and one from the set \{5, 7, 13, 17\}. Thus the problem of classification of all simple groups of order $r^a q^b p^c$, $r$, $q$ and $p$ primes, reduces to the classification of simple groups of order mentioned in the title.

In [1] Brauer solved this problem for the case $p^c = 5$. The author showed in [5] that if one of the Sylow groups of $G$ is cyclic and if it is not the Sylow subgroup of $G$ of least order, then $G$ is isomorphic to one of the groups: $PSL(2, 5)$, $PSL(2, 7)$, $PSL(2, 8)$ and $PSL(2, 17)$. The purpose of this paper is to classify all simple groups of order $2^a 3^b p^c$ with a cyclic Sylow 3-subgroup.

**Theorem 1.** Let $G$ be a simple group of order $2^a 3^b p^c$, where $p$ is a prime, and suppose that a Sylow 3-subgroup $Q$ of $G$ is cyclic. Then $G$ is isomorphic to one of the groups: $PSL(2, 5)$, $PSL(2, 7)$, $PSL(2, 8)$ and $PSL(2, 17)$.

As a matter of fact, we will prove the following more general result:

**Theorem 2.** Let $G$ be a simple group of order $2^a 3^b p^c$, where $p$ is a prime. Suppose that a Sylow subgroup $R$ of $G$ is cyclic and $[N_G(R) : C_G(R)] = 2$. Then $G$ is isomorphic to one of the groups: $PSL(2, 5)$, $PSL(2, 7)$, $PSL(2, 8)$, $PSL(2, 9)$ and $PSL(2, 17)$.

Theorem 1 follows easily from Theorem 2. Indeed, it is well known that $[N_G(Q) : C_G(Q)] = 1$ or 2. It follows from the simplicity of $G$ and the theorem of Burnside that $[N_G(Q) : C_G(Q)] = 2$. Thus the assumptions of Theorem 2 are satisfied and it is well known that the groups mentioned in Theorem 2, $PSL(2, 9)$ excluded, satisfy the assumptions of Theorem 1.

**Proof of Theorem 2.** Let $R$ be a Sylow $r$-subgroup of $G$ of order $r^b$. Since $G$ is simple $r \neq 2$ and $2^a \geq 4$. It follows from Proposition 2.1 and Corollary 2.1 in [4] that the principal $r$-block $B_1$ of $G$ contains $(r^b - 1)/2$ exceptional characters of order $x_0 = a_0 r^b - 2 e_0$ and two non-exceptional (ordinary) characters: the principal character $1_G$ and

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another character $X_2$ of order $x_2 = a_2 r^5 + \epsilon_2$, where $a_0, a_2$ are nonnegative integers and $\epsilon_0, \epsilon_2 = \pm 1$. Formula (6) in [4] then yields:

$$0 = 1 + \epsilon_0 x_0 + \epsilon_2 x_2 = 1 + \epsilon_0 a_0 r^5 - 2 + \epsilon_2 a_2 r^5 + 1.$$  

Therefore $a_0 = a_2$, $\epsilon_2 = -\epsilon_0$ and consequently

$$x_0 = a_0 r^5 - 2\epsilon_0, \quad x_2 = a_0 r^5 - \epsilon_0.$$  

Thus $x_0$, $r$ and $x_2$ are prime to each other, and it follows from our assumptions that one of the following cases holds:

**Case A:** $x_0 = 2^r$, $x_2 = u^r$,

**Case B:** $x_0 = u^r$, $x_2 = 2^r$,

where \{u, r\} = \{3, p\} and $\nu, \eta$ are positive integers. In both cases it follows from the formulas for $x_0$ and $x_2$ that

$$u^r - 2^r = \epsilon, \quad \epsilon = \pm 1.$$  

If $\eta = 1$ then it is well known [3, Theorem 18.4] that $u$ is the order of a Sylow $u$-subgroup of $G$. Consequently, all the Sylow subgroups of odd order of $G$ are cyclic and by Theorem 1 of [4] $G$ is isomorphic to one of the groups: $\text{PSL}(2, 5)$, $\text{PSL}(2, 7)$, $\text{PSL}(2, 8)$ and $\text{PSL}(2, 17)$. Since these groups satisfy the assumptions of Theorem 2, we are done in the case $\eta = 1$.

Now assume that $\eta > 1$. Then by [6, Theorem 2, p. 335 and Exercise 1, p. 346] $u = 3, \eta = 2, \nu = 3$ and $\epsilon = 1$. We will deal now separately with Cases A and B.

**Case A.** It follows from the formulas for $x_0$ and $x_2$ that:

$$8 = x_0 = a_0 r^5 - 2\epsilon_0, \quad 9 = x_2 = x_0 + \epsilon_0$$  

hence: $r^5 = 5$. It follows then from [1] that $G$ is isomorphic either to $\text{PSL}(2, 5)$ or to $\text{PSL}(2, 9)$. Since the order of $\text{PSL}(2, 5)$ is not divisible by 9, only $\text{PSL}(2, 9)$ satisfies the assumptions of Case A.

**Case B.** The formulas for $x_0$ and $x_2$ now yield:

$$9 = x_0 = a_0 r^5 - 2\epsilon_0, \quad 8 = x_2 = x_0 + \epsilon_0$$  

hence $r^5 = 7$. Thus $B_1$ contains the irreducible character $X_2$ of degree $8 < 2r = 14$ and by Lemma 1 of [2] $C_G(\rho) = R$ for every nonidentity element $\rho$ of $R$. Consequently, every $r$-singular element of $G$ is of order $r$. Lemma 3 of [2] then yields that if $B_2$ is the 2-block to which $X_2$ belongs then:

$$\sum X(1) X(\rho) \equiv 0 \pmod{2^n} \quad X \text{ in } B_1 \cap B_2$$  

where $\rho$ is any $r$-singular element of $G$. Since by Lemma 2 of [2] $B_1$
is a block of defect $\alpha - 1$ at most, it contains characters of even orders only and therefore $B_1 \cap B_2 = \{ X_2 \}$. As $\rho$ is conjugate to an element of $R^\sharp$, it follows from Proposition 2.1 of [4] that
\[ X_2(\rho) = \epsilon_2 = - \epsilon_0 = 1. \]
The above summation formula then reads:
\[ s = x_2 \cdot 1 \equiv 0 \pmod{2^\alpha}. \]
Since $x_2$ divides $o(G)$, the order of $G$, it follows that $2^\alpha = 8$ and
\[ o(G) = 7 \cdot 8 \cdot 3^\beta, \quad \beta \geq 2. \]

Let $T$ be a Sylow 2-subgroup of $G$. It is well known that $T$ cannot be the quaternion group. If $T$ is dihedral or Abelian, then by Theorem 2 in [5] $G$ has to be isomorphic to one of the groups $\text{PSL}(2, 5)$, $\text{PSL}(2, 7)$, $\text{PSL}(2, 8)$, $\text{PSL}(2, 9)$ and $\text{PSL}(2, 17)$. It is easy to check that only $\text{PSL}(2, 8)$ has order of the form $7 \cdot 8 \cdot 3^\beta$, $\beta \geq 2$, and consequently only $\text{PSL}(2, 8)$ satisfies the assumptions of Case B. The proof of Theorem 2 is complete.

References

1. R. Brauer, On simple groups of order $5 \cdot 3^a \cdot 2^2$, Bull. Amer. Math. Soc. 74 (1968), 900-903.
4. M. Herzog, On finite groups with cyclic Sylow subgroups for all odd primes, Israel J. Math. 6 (1968), 206-216.
5. ———, On finite simple groups of order divisible by three primes only, J. Algebra 10 (1968), 383-388.
7. J. G. Thompson, Nonsolvable finite groups all of whose local subgroups are solvable, Sec. 1-6, Bull. Amer. Math. Soc. 74 (1968), 383-437 (balance to appear).