RECURRENCE AND PRESERVATION OF MEASURE

F. H. SIMONS

Let $T$ be a measurable (not necessarily invertible) transformation in a (not necessarily σ-finite) measure space $(X, \mathcal{R}, \mu)$. The following theorem extends results of Halmos-Ornstein [1] and Helmberg [2], while the proof is rather simple.

**Theorem.** Suppose $\mu(T^{-1}A) \leq \mu(A)$ for all $A \in \mathcal{R}$. If $E \in \mathcal{R}$ satisfies $E \subseteq \bigcup_{n=1}^{\infty} T^{-n}E[\mu]$, then $\mu(T^{-1}E) = \mu(E)$.

**Proof.** Suppose $\mu(T^{-1}E) < \mu(E)$. Put $E_1 = T^{-1}E \cap E$ and $A_1 = T^{-1}E \cap (X \setminus E)$, then $E_1 \in \mathcal{R}$, $A_1 \in \mathcal{R}$, and $\mu(E) > \mu(E_1) + \mu(A_1)$.

The sets $E_n \in \mathcal{R}$, $A_n \in \mathcal{R}$ are defined inductively for $n \geq 2$ by

\[
E_n = T^{-1}A_{n-1} \cap E, \quad A_n = T^{-1}A_{n-1} \cap (X \setminus E).
\]

It follows that for every $n \geq 2$

\[
\mu(A_1) \geq \mu(E_2) + \cdots + \mu(E_n) + \mu(A_n)
\]

hence

\[
\mu(A_1) \geq \sum_{n=2}^{\infty} \mu(E_n), \quad \mu(E) > \sum_{n=1}^{\infty} \mu(E_n).
\]

Since $\bigcup_{n=1}^{\infty} E_n$ is the set of recurrent points of $E$, we must have $\mu(E \setminus \bigcup_{n=1}^{\infty} E_n) = 0$. Contradiction.

In particular, if $T$ is also conservative, then $E \subseteq \bigcup_{n=1}^{\infty} T^{-n}E[\mu]$ for all $E \in \mathcal{R}$, hence $T$ is measure preserving.

As a corollary we obtain by repeating the proof given in [2, §4], that if $T$ is a conservative measure preserving transformation in $(X, \mathcal{R}, \mu)$, for every $E \in \mathcal{R}$ the induced transformation $T_E$ on $(E, \mathcal{R} \cap E, \mu)$ is conservative and measure preserving.

**References**


**Technological University, Eindhoven, Netherlands**

Received by the editors May 10, 1968.