RECURRENCE AND PRESERVATION OF MEASURE

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Let T be a measurable (not necessarily invertible) transformation in a (not necessarily σ-finite) measure space $(X, \mathfrak{R}, \mu)$. The following theorem extends results of Halmos-Ornstein [1] and Helmberg [2], while the proof is rather simple.

THEOREM. *Suppose* $\mu(T^{-1}A) \leqq \mu(A)$ *for all* $A \in \mathfrak{R}$. *If* $E \in \mathfrak{R}$ *satisfies* $E \subset \bigcup_{n=1}^{\infty} T^{-n}E[\mu]$, *then* $\mu(T^{-1}E) = \mu(E)$.

PROOF. Suppose $\mu(T^{-1}E) < \mu(E)$. Put $E_1 = T^{-1}E \cap E$ and $A_1 = T^{-1}E \cap (X \setminus E)$, then $E_1 \in \mathfrak{R}$, $A_1 \in \mathfrak{R}$, and

$$\mu(E) > \mu(E_1) + \mu(A_1).$$

The sets $E_n \in \mathfrak{R}$, $A_n \in \mathfrak{R}$ are defined inductively for $n \geqq 2$ by

$$E_n = T^{-1}A_{n-1} \cap E, \qquad A_n = T^{-1}A_{n-1} \cap (X \setminus E).$$

It follows that for every $n \geqq 2$

$$\mu(A_1) \geqq \mu(E_2) + \cdots + \mu(E_n) + \mu(A_n)$$

hence

$$\mu(A_1) \geqq \sum_{n=2}^{\infty} \mu(E_n), \qquad \mu(E) > \sum_{n=1}^{\infty} \mu(E_n).$$

Since $\bigcup_{n=1}^{\infty} E_n$ is the set of recurrent points of E, we must have $\mu(E \setminus \bigcup_{n=1}^{\infty} E_n) = 0$. Contradiction.

In particular, if T is also conservative, then $E \subset \bigcup_{n=1}^{\infty} T^{-n}E[\mu]$ for all $E \in \mathfrak{R}$, hence T is measure preserving.

As a corollary we obtain by repeating the proof given in [2, §4], that if T is a conservative measure preserving transformation in $(X, \mathfrak{R}, \mu)$, for every $E \in \mathfrak{R}$ the induced transformation T_E on $(E, \mathfrak{R} \cap E, \mu)$ is conservative and measure preserving.

REFERENCES

1. P. R. Halmos, *Lectures on ergodic theory*, Publ. Math. Soc. Japan, no. 3, The Mathematical Society of Japan and Chelsea, New York, 1960. MR **20** #3958, MR **22** #2677.

2. G. Helmberg, *Über konservative Transformationen*, Math. Ann. **165** (1966), 44–61. MR **33** #5842.

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